Relaxation of Solitons in Nonlinear Schrödinger Equations with Potential*

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Abstract

In this paper we study dynamics of solitons in the generalized nonlinear Schrödinger equation (NLS) with an external potential in all dimensions except for 2. For a certain class of nonlinearities such an equation has solutions which are periodic in time and exponentially decaying in space, centered near different critical points of the potential. We call those solutions which are centered near the minima of the potential and which minimize energy restricted to $\mathcal{L}^2-$unit sphere, *trapped solitons* or just *solitons*.

In this paper we prove, under certain conditions on the potentials and initial conditions, that trapped solitons are asymptotically stable. Moreover, if an initial condition is close to a trapped soliton then the solution looks like a moving soliton relaxing to its equilibrium position. The dynamical law of motion of the soliton (i.e. effective equations of motion for the soliton's center and momentum) is close to Newton's equation but with a dissipative term due to radiation of the energy to infinity.

1 Introduction

In this paper we study dynamics of solitons in the generalized nonlinear Schrödinger equation (NLS) in dimension $d \neq 2$ with an external potential $V_h : \mathbb{R}^n \to \mathbb{R}$,

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + V_h\psi - f(|\psi|^2)\psi. \tag{1}$$

Here $h > 0$ is a small parameter giving the length scale of the external potential in relation to the length scale of the $V_h = 0$ solitons (see below), Δ is the Laplace operator and $f(s)$ is a nonlinearity to be specified later. We normalize $f(0) = 0$.

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Such equations arise in the theory of Bose-Einstein condensation [1], nonlinear optics, theory of water waves [2]and in other areas.

To fix ideas we assume the potentials to be of the form $V_h(x) := V(hx)$ with V smooth and decaying at ∞. Thus for $h = 0$, Equation (1) becomes the standard generalized nonlinear Schrödinger equation (gNLS)

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + \mu\psi - f(|\psi|^2)\psi, \tag{2}$$

where $\mu = V(0)$. For a certain class of nonlinearities, $f(|\psi|^2)$ (see Section 3), there is an interval $\mathcal{I}_0 \subset \mathbb{R}^n$ such that for any $\lambda \in \mathcal{I}_0$ Equation (2) has solutions of the form $e^{i(\lambda-\mu)t}\phi_0^\lambda(x)$ where $\phi_0^\lambda \in \mathcal{H}_2(\mathbb{R})$ and $\phi_0^\lambda > 0$. Such solutions (in general without the restriction $\phi_0^\lambda > 0$) are called the *solitary waves* or *solitons* or, to emphasize the property $\phi_0^\lambda > 0$, the *ground states*. For brevity we will use the term *soliton* applying it also to the function ϕ_0^λ without the phase factor $e^{i(\lambda-\mu)t}$.

Equation (2) is translationally and gauge invariant. Hence if $e^{i(\lambda-\mu)t}\phi_0^\lambda(x)$ is a solution for Equation (2), then so is

$$e^{i(\lambda-\mu)t}e^{i\alpha}\phi_0^\lambda(x+a),\ \text{for any } a \in \mathbb{R}^n, \text{and } \alpha \in [0, 2\pi).$$

This situation changes dramatically when the potential V_h is turned on. In general, as was shown in [FW, Oh1, ABC] out of the $n+2$-parameter family $e^{i(\lambda-\mu)t}e^{i\alpha}\phi_0^\lambda(x+a)$ only a discrete set of two parameter families of solutions to Equation (1) bifurcate: $e^{i\lambda t}e^{i\alpha}\phi^\lambda(x)$, $\alpha \in [0, 2\pi)$ and $\lambda \in \mathcal{I}$ for some $\mathcal{I} \subseteq \mathcal{I}_0$, with $\phi^\lambda \equiv \phi_h^\lambda \in \mathcal{H}_2(\mathbb{R})$ and $\phi^\lambda > 0$. Each such family centers near a different critical point of the potential $V_h(x)$. It was shown in [Oh2] that the solutions corresponding to minima of $V_h(x)$ are orbitally (Lyapunov) stable and to maxima, orbitally unstable. We call the solitary wave solutions described above which correspond to the minima of $V_h(x)$ *trapped solitons* or just *solitons* of Equation (1) omitting the last qualifier if it is clear which equation we are dealing with.

The main result of this paper is a proof that the trapped solitons of Equation (1) are asymptotically stable. The latter property means that if an initial condition of (1) is sufficiently close to a trapped soliton then the solution converges (relaxes),

$$\psi(x,t) - e^{i\gamma(t)}\phi^{\lambda_\infty} \to 0,$$

in some weighted $\mathcal{L}^2$ space to, in general, another trapped soliton of the same two-parameter family. We also find effective equations for the soliton center and other parameters. In this paper we prove this result under the additional assumption that if $d > 2$ then the potential is spherically symmetric and that the initial condition symmetric with respect to permutations of the coordinates. In this case the soliton relaxes to the ground state along the radial direction.

[1]In this case Equation (1) is called the Gross-Pitaevskii equation.

[2]In these two areas V_h arises if one takes into account impurities and/or variations in geometry of the medium and is, in general, time-dependent.

This limits the number of technical difficulties we have to deal with. We expect that our techniques extend to the general case when the soliton spirals toward its equilibrium.

In fact, we prove a result more general than asymptotic stability of trapped solitons. Namely, we show that if an initial condition is close (in the weighted norm $\|u\|_{\nu,1} := \|(1+|x|^2)^{-\frac{\nu}{2}}u\|_{\mathcal{H}^1}$ for sufficiently large ν) to the soliton $e^{i\gamma_0}\phi^{\lambda_0}$, with $\gamma_0 \in \mathbb{R}$ and $\lambda_0 \in \mathcal{I}$ ($\mathcal{I}$ as above), then the solution, $\psi(t)$, of Equation (1) can be written as

$$\psi(x,t) = e^{i\gamma(t)}\big(e^{ip(t)\cdot x}\phi^{\lambda(t)}(x-a(t)) + R(x,t)\big), \tag{3}$$

where $\|R(t)\|_{\nu,1} \to 0$, $\lambda(t) \to \lambda_\infty$ for some λ_∞ as $t \to \infty$ and the soliton center $a(t)$ and momentum $p(t)$ evolve according to an effective equations of motion close to Newton's equation in the potential $h^2V(a)$.

We observe that (1) is a Hamiltonian system with conserved energy (see Section 2) and, though orbital (Lyapunov) stability is expected, the asymptotic stability is a subtle matter. To have asymptotic stability the system should be able to dispose of excess of its energy, in our case, by radiating it to infinity. The infinite dimensionality of a Hamiltonian system in question plays a crucial role here. This phenomenon as well as a general class of classical and quantum relaxation problems was pointed out by J. Fröhlich and T. Spencer [FS].

We also mention that because of slow time-decay of the linearized propagator, the low dimensions $d = 1, 2$ are harder to handle than the higher dimensions, $d > 2$.

We refer to [GS1] for a detailed review of the related literature. Here we only mention results of [Cu, BP1, BP2, BuSu, SW1, SW2, SW3, SW4, TY1, TY2, TY3] which deal with a similar problem. Like our work, [SW1, SW2, SW3, SW4, TY1, TY2, TY3] study the ground state of the NLS with a potential. However, these papers deal with the near-linear regime in which the nonlinear ground state is a bifurcation of the ground state for the corresponding Schrödinger operator $-\Delta + V(x)$. The present paper covers highly nonlinear regime in which the ground state is produced by the nonlinearity (our analysis simplifies considerably in the near-linear case). Now, papers [Cu, BP1, BP2, BuSu] consider the NLS without a potential so the corresponding solitons, which were described above, are affected only by a perturbation of the initial conditions which disperses with time leaving them free. While in our case they, in addition, are under the influence of the potential and they relax to an equilibrium state near a local minimum of the potential.

We formulate some open problem:

(1) Extend the results of this present paper to more general initial conditions and to more general, probably time-dependent, potentials.

(2) Think the results of this paper with the results of [FGJS] on the long time dynamics of solitons.

A natural place to start here is spherically symmetric potentials but general initial conditions. Note that for certain time-dependent potentials the solitons will never settle in the ground state.

As customary we often denote derivatives by subindices as in $\phi^\lambda_\lambda = \frac{\partial}{\partial\lambda}\phi^\lambda$ for $\phi^\lambda = \phi^\lambda(x)$. However, the subindex h signifies always the dependence on the parameter h and not the derivatives in h. The Sobolev and L^2 spaces are denoted by $\mathcal{H}^k$ and $\mathcal{L}^2$ respectively.

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2 Hamiltonian Structure and GWP

Equation (1) is a Hamiltonian system on Sobolev space $\mathcal{H}^1(\mathbb{R},\mathbb{C})$ viewed as a real space $\mathcal{H}^1(\mathbb{R},\mathbb{R}) \oplus \mathcal{H}^1(\mathbb{R},\mathbb{R})$ with the inner product $(\psi,\phi) = Re\int_{\mathbb{R}} \bar{\psi}\phi$ and with the simpletic form $\omega(\psi,\phi) = Im\int_{\mathbb{R}} \bar{\psi}\phi$. The Hamiltonian functional is:

$$H(\psi) := \int [\frac{1}{2}(|\psi_x|^2 + V_h|\psi|^2) - F(|\psi|^2)],$$

where $F(u) := \frac{1}{2}\int_0^u f(\xi)d\xi$.

Equation (1) has the time-translational and gauge symmetries which imply the following conservation laws: for any $t \geq 0$, we have

(CE) conservation of energy: $H(\psi(t)) = H(\psi(0))$;

(CP) conservation of the number of particles: $N(\psi(t)) = N(\psi(0))$, where $N(\psi) := \int |\psi|^2$.

To address the global well-posedness of (1) we need the following condition on the nonlinearity f.

(fA) The nonlinearity f satisfies the estimate

$$|f^{'}(\xi)| \leq c(1 + |\xi|^{\alpha-1})$$

for some $\alpha \in [0, \frac{2}{(d-2)_+})$ (here $s_+ = s$ if $s > 0$ and $= 0$ if $s \leq 0$) and

$$|f(\xi)| \leq c(1 + |\xi|^{\beta})$$

for some $\beta \in [0, \frac{2}{d})$.

The following theorem is proved in [Oh3, Caz].

Theorem *Assume that the nonlinearity f satisfies the condition (fA), and that the potential V is bounded. Then Equation (1) is globally well posed in $\mathcal{H}^1$, i.e. the Cauchy problem for Equation (1) with a datum $\psi(0) \in \mathcal{H}^1$ has a unique solution $\psi(t)$ in the space $\mathcal{H}^1$ and this solution depends continuously on $\psi(0)$. Moreover $\psi(t)$ satisfies the conservation laws (CE) and (CP).*

3 Existence and Orbital Stability of Solitons

In this section we review the question of existence of the solitons (ground states) for Equation (1). Assume the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies

(fB) There exists an interval $\mathcal{I}_0 \in \mathbb{R}^+$ s.t. for any $\lambda \in \mathcal{I}_0$, $-\infty \leq \overline{\lim}_{s\to+\infty} \frac{f(s)}{s^{\frac{2}{d-2}}} \leq 0$ and $\frac{1}{\xi}\int_0^\xi f(s)ds > \lambda$ for some constant ξ, for $d > 2$; and

$$U(\phi,\lambda) := -\lambda\phi^2 + \int_0^{\phi^2} f(\xi)d\xi$$

has a smallest positive root $\phi_0(\lambda)$ such that $U_\phi(\phi_0(\lambda),\lambda) \neq 0$, for $d = 1$.

It is shown in [BL, Strauss] that under Condition (fB) there exists a spherical symmetric positive solution ϕ^λ to the equation

$$-\Delta\phi^\lambda + \lambda\phi^\lambda - f((\phi^\lambda)^2)\phi^\lambda = 0. \tag{4}$$

Remark 1. *Existence of soliton functions ϕ^λ for $d = 2$ is proved in [Strauss] under different conditions on f.*

When the potential V is present, then some of the solitons above bifurcate into solitons for Equation (1). Namely, let, in addition, f satisfy the condition $|f'(\xi)| \leq c(1 + |\xi|^p)$, for some $p < \infty$, and V satisfy the condition

(VA) V is smooth and 0 is a non-degenerate local minimum of V.

Then, similarly as in [FW, Oh1, ABC] one can show that if h is sufficiently small, then for any $\lambda \in \mathcal{I}_{0V}$, where

$$\mathcal{I}_{0V} := \{\lambda | \lambda > -\inf_{x\in\mathbb{R}}\{V(x)\}\} \cap \{\lambda | \lambda + V(0) \in \mathcal{I}_0\},$$

there exists a unique soliton $\phi^\lambda \equiv \phi_h^\lambda$ (i. e. $\phi^\lambda \in \mathcal{H}_2(\mathbb{R})$ and $\phi^\lambda > 0$) satisfying the equation

$$-\Delta\phi^\lambda + (\lambda + V_h)\phi^\lambda - f((\phi^\lambda)^2)\phi^\lambda = 0$$

and the estimate $\phi^\lambda = \phi_0^{\lambda+V(0)} + O(h^{3/2})$ where ϕ_0^λ is the soliton of Equation (4).

Let $\delta(\lambda) := \|\phi^\lambda\|_2^2$. It is shown in [GSS1] that the soliton ϕ^λ is a minimizer of the energy functional $H(\psi)$ for a fixed number of particles $N(\psi) = \textit{constant}$ if and only if

$$\delta'(\lambda) > 0. \tag{5}$$

Moreover, it shown in [We2, GSS1] that under the latter condition the solitary wave $\phi^\lambda e^{i\lambda t}$ is orbitally stable. Under more restrictive conditions (see [GSS1]) on f one can show that the open set

$$\mathcal{I} := \{\lambda \in \mathcal{I}_{0V} : \delta'(\lambda) > 0\} \tag{6}$$

is non-empty. Instead of formulating these conditions we assume in what follows that the open set $\mathcal{I}$ is non-empty and $\lambda \in \mathcal{I}$.

Using the equation for ϕ^λ one can show that if the potential V is redially symmetric then there exist constants $c, \ \delta > 0$ such that

$$|\phi^\lambda(x)| \leq ce^{-\delta|x|} \text{ and } |\frac{d}{d\lambda}\phi^\lambda| \leq ce^{-\delta|x|}, \tag{7}$$

and similarly for the derivatives of ϕ^λ and $\frac{d}{d\lambda}\phi^\lambda$.

4 Linearized Equation and Resonances

We rewrite Equation (1) as $\frac{d\psi}{dt} = G(\psi)$ where the nonlinear map $G(\psi)$ is defined by

$$G(\psi) = -i(-\Delta + \lambda + V_h)\psi + if(|\psi|^2)\psi. \tag{8}$$

Then the linearization of Equation (1) can be written as $\frac{\partial\chi}{\partial t} = \partial G(\phi^\lambda)\chi$ where $\partial G(\phi^\lambda)$ is the Fréchet derivative of $G(\psi)$ at ϕ. It is computed to be

$$\partial G(\phi^\lambda)\chi = -i(-\Delta + \lambda + V_h)\chi + if((\phi^\lambda)^2)\chi + 2if^{'}((\phi^\lambda)^2)(\phi^\lambda)^2 Re\chi. \tag{9}$$

This is a real linear but not complex linear operator. To convert it to a linear operator we pass from complex functions to real vector-functions

$$\chi \longleftrightarrow \vec{\chi} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix},$$

where $\chi_1 = Re\chi$ and $\chi_2 = Im\chi$. Then $\partial G(\phi^\lambda)\chi \longleftrightarrow L(\lambda)\vec{\chi}$ where the operator $L(\lambda)$ is given by

$$L(\lambda) := \begin{pmatrix} 0 & L_-(\lambda) \\ -L_+(\lambda) & 0 \end{pmatrix}, \tag{10}$$

with

$$L_-(\lambda) := -\Delta + V_h + \lambda - f((\phi^\lambda)^2), \tag{11}$$

and

$$L_+(\lambda) := -\Delta + V_h + \lambda - f((\phi^\lambda)^2) - 2f^{'}((\phi^\lambda)^2)(\phi^\lambda)^2. \tag{12}$$

Then we extend the operator $L(\lambda)$ to the complex space $\mathcal{H}^2(\mathbb{R}, \mathbb{C}) \oplus \mathcal{H}^2(\mathbb{R}, \mathbb{C})$.

By a general result (see e.g. [HS, RSIV]),

$$\sigma_{ess}(L(\lambda)) = (-i\infty, -i\lambda] \cap [i\lambda, i\infty)$$

if the potential V_h in Equation (1) decays at ∞.

The eigenfunctions of $L(\lambda)$ are described in the following theorem (cf [GS1]).

Theorem 4.1. *Let V satisfy Condition (VA). Then the operator $L(\lambda)$ has at least $2d + 2$ eigenvectors and associated eigenvectors with eigenvalues near near zero: two-dimensional space with the eigenvalue 0 and a $2d$-dimensional space with non-zero imaginary eigenvalues.*

Proof. We already know that the vector $\begin{pmatrix} 0 \\ \phi^\lambda \end{pmatrix}$ is an eigenvector of $L(\lambda)$ with eigenvalue 0 and $\begin{pmatrix} \partial_\lambda \phi^\lambda \\ 0 \end{pmatrix}$ is an associated eigenvector,

$$L(\lambda) \begin{pmatrix} 0 \\ \phi^\lambda \end{pmatrix} = 0, \; L(\lambda) \begin{pmatrix} \partial_\lambda \phi^\lambda \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi^\lambda \end{pmatrix}. \tag{13}$$

Similarly as in [GS1] one can show that the operator $L(\lambda)$ has also the eigenvalues $\pm i\epsilon_j(\lambda)$, $\epsilon_j(\lambda) > 0$, with the eigenfunctions $\begin{pmatrix} \xi_j \\ \pm i\eta_j \end{pmatrix}$, related by complex conjugation. Moreover, $\epsilon_j(\lambda) := h\sqrt{2e_j} + o(h)$ where e_j are eigenvalues of the Hessian matrix of V at value $x = 0$, $V^{''}(0)$, and

$$\xi_j = \sqrt{2}\partial_{x_k}\phi_0^\lambda + o(h) \text{ and } \eta_j = -h\sqrt{e_j}x_j\phi_0^\lambda + o(h),$$

and ξ_i and η_j are real. □

Remark 2. *The zero eigenvector* $\begin{pmatrix} 0 \\ \phi^\lambda \end{pmatrix}$ *and the associated zero eigenvector* $\begin{pmatrix} \partial_\lambda \phi^\lambda \\ 0 \end{pmatrix}$ *are related to the gauge symmetry* $\psi(x,t) \to e^{i\alpha}\psi(x,t)$ *of the original equation and the* $2d$ *eigenvectors* $\begin{pmatrix} \xi_j \\ \pm i\eta_j \end{pmatrix}$ *with* $O(h)$ *eigenvalues originate from the zero eigenvectors* $\begin{pmatrix} \partial_{x_k}\phi_0^\lambda \\ 0 \end{pmatrix}$, $k = 1, 2, \cdots, d$, *and the associated zero eigenvectors* $\begin{pmatrix} 0 \\ x_k\phi_0^\lambda \end{pmatrix}$, $k = 1, 2, \cdots, d$, *of the* $V = 0$ *equation due to the translational symmetry and to the boost transformation* $\psi(x,t) \to e^{ib\cdot x}\psi(x,t)$ *(coming from the Galilean symmetry), respectively.*

We say that a function $g \in \mathcal{L}^2(\mathbb{R}^d)$ is permutational symmetric if

$$g(x) = g(\sigma x) \text{ for any } \sigma \in S_d$$

with S_d being the group of permutation of d indices and

$$\sigma(x_1, x_2, \cdots, x_d) := (x_{\sigma(1)}, x_{\sigma(2)}, \cdots, x_{\sigma(d)}).$$

Remark 3. *For any function of the form* $e^{ip\cdot x}\phi(|x-a|)$ *with* $a \parallel p$, *there exists a rotation* τ *such that the function* $e^{ip\cdot\tau x}\phi(|\tau x - a|) = e^{i\tau^{-1}p\cdot x}\phi(|x - \tau^{-1}a|)$ *is permutational symmetric. Such families describe wave packets with the momenta directed toward or away from the origin.*

If for $d \geq 2$ the potential $V(x)$ is spherically symmetric, then $V^{''}(0) = \frac{1}{d}\Delta V(0) \cdot Id_{n\times n}$, and therefore the eigenvalues e_j of $V^{''}(0)$ are all equal to $\frac{1}{d}\Delta V(0)$. Thus we have

Corollary 4.2. *Let $d \geq 2$ and V satisfy Condition (VA) and let V be spherically symmetric. Then $L(\lambda)$ restricted to permutational symmetric functions has 4 eigenvectors or associated eigenvectors near zero: two-dimensional space with eigenvalue 0; and two-dimensional space with the non-zero imaginary eigenvalues $\pm i\epsilon(\lambda)$, where $\epsilon(\lambda) = h\sqrt{\frac{2\Delta V(0)}{d}} + o(h)$, and with the eigenfunctions $\begin{pmatrix} \xi \\ \pm i\eta \end{pmatrix}$, where ξ and η are real, and permutation symmetric functions satisfying*

$$\xi = \sqrt{2}\sum_{n=1}^{d} \frac{d}{dx_n}\phi_0^\lambda + O(h) \text{ and } \eta = -h\sqrt{\frac{1}{d}\Delta V(0)}\sum_{n=1}^{d} x_n \phi_0^\lambda + O(h^{3/2}).$$

The eigenvectors $\begin{pmatrix} \xi \\ \pm i\eta \end{pmatrix}$ are symmetric combinations of the eigenvectors described in the proof of Theorem 2. Observe that

$$Span\{\phi^\lambda, \phi_\lambda^\lambda\} \perp Span\{\xi, \eta\} \tag{14}$$

since $\phi^\lambda, \phi_\lambda^\lambda$ are spherically symmetric, and

$$\langle \xi, \eta \rangle = \frac{1}{\epsilon(\lambda)}\langle L_-(\lambda)\eta, \eta\rangle > 0. \tag{15}$$

Besides eigenvalues, the operator $L(\lambda)$ may have resonances at the tips, $\pm i\lambda$, of its essential spectrum (those tips are called thresholds). To define the resonance we write the operator $L(\lambda)$ as $L(\lambda) = L_0(\lambda) + V_{big}(\lambda)$, where

$$L_0(\lambda) := \begin{pmatrix} 0 & -\Delta + \lambda \\ \Delta - \lambda & 0 \end{pmatrix}, \tag{16}$$

and

$$V_{big}(\lambda) := \begin{pmatrix} 0 & V_h - f((\phi^\lambda)^2) \\ -V_h + f((\phi^\lambda)^2) + 2f'((\phi^\lambda)^2)(\phi^\lambda)^2 & 0 \end{pmatrix}. \tag{17}$$

Recall the notation $\alpha_+ := \alpha$ if $\alpha > 0$ and $= 0$ of $\alpha \leq 0$.

Definition 4.3. *Let $d \neq 2$. A function h is called a resonance function of $L(\lambda)$ at $\mu = \pm i\lambda$ if $h \notin \mathcal{L}^2$, $|h(x)| \leq c\langle x\rangle^{-(d-2)_+}$ and h is C^2 and solves the equation*

$$(L(\lambda) - \mu)h = 0.$$

Note that this definition implies that for $d > 2$ the resonance function h solves the equation

$$(1 + K(\lambda))h = 0$$

where $K(\lambda)$ is a family of compact operators given by $K(\lambda) := (L_0(\lambda) - \mu + 0)^{-1}V_{big}(\lambda)$.

In this paper we make the following assumptions for the point spectrum and resonances of the operator $L(\lambda)$:

(SA) $L(\lambda)$ has only 4 standard and associated eigenvectors in the permutation symmetric subspace.

(SB) $L(\lambda)$ has no resonances at $\pm i\lambda$.

The discussion and results concerning these conditions, given in [GS1], suggested strongly that Condition (SA) is satisfied for a large class of nonlinearities and potentials and Condition (SB) is satisfied generically. In [GSV] we show this using earlier results of [CP, CPV]. We also assume the following condition

(FGR) Let N be the smallest positive integer such that $\epsilon(\lambda)(N+1) > \lambda\ \forall \lambda \in I$. Then $ReY_N < 0$ where $Y_n,\ n = 1, 2, \cdots$, are the functions of V and λ, defined in Lemma 8.3 below.

We expect that Condition (FGR) holds generically. Theorem 5.2 below shows that $ReY_n = 0$ if $n < N$.

We expect the following is true: (a) if for some $N_1(\geq N)$, $ReY_n = 0$ for $n < N_1$, then $ReY_{N_1} \leq 0$ and (b) for generic potentials and nonlinearities there exists an $N_1(\geq N)$ such that $ReY_{N_1} \neq 0$. Thus Condition (FGR) could have been generalized by assuming that $ReY_{N_1} < 0$ for some $N_1 \geq N$ such that $ReY_n = 0$ for $n < N_1$. We took $N = N_1$ in order not to complicate the exposition.

The following form of ReY_N

$$ReY_N = Im\langle \sigma_1(L(\lambda) - (N+1)i\epsilon(\lambda) - 0)^{-1}F, F\rangle \leq 0 \tag{18}$$

for some function F depending on λ and V and the matrix $\sigma_1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, is proved in [BuSu, TY1, TY2, SW] for $N = 1$, and in [G] for $N = 2, 3$. We conjecture that this formula holds for any N.

Condition (FGR) is related to the Fermi Golden Rule condition which appears whenever time-(quasi)periodic, spatially localized solutions become coupled to radiation. In the standard case it says that this coupling is effective in the second order ($N = 1$) of the perturbation theory and therefore it leads to instability of such solutions. In our case these time-periodic solutions are stationary solutions

$$c_1 \begin{pmatrix} \xi \\ i\eta \end{pmatrix} e^{i\epsilon(\lambda)t} + c_2 \begin{pmatrix} \xi \\ -i\eta \end{pmatrix} e^{-i\epsilon(\lambda)t}$$

of the linearized equation $\frac{\partial \vec{\chi}}{\partial t} = L(\lambda)\vec{\chi}$ and the coupling is realized through the nonlinearity. Since the radiation in our case is "massive"$-$ the essential spectrum of $L(\lambda)$ has the gap $(-i\lambda, i\lambda)$, $\lambda > 0$, $-$ the coupling occurs only in the $N-$th order of perturbation theory where N is the same as in Condition (FGR).

The rigorous form of the Fermi Golden Rule for the linear Schrödinger equations was introduced in [Simon] (see [RSIV]). For nonlinear waves and Schrödinger equations the Fermi Golden Rule and the corresponding condition were introduced in [S] and, in the present context, in [CLR, SW, BuSu, BP2, TY1, TY2, TY3].

5 Main Results

In this section we state the main theorem of this paper. For technical reason we impose the following conditions on f and V

(fC) the nonlinearity f is a smooth function satisfying $f^{''}(0) = f^{'''}(0) = 0$ if $d \geq 3$; and $f^{(k)}(0) = 0$ for $k = 2, 3 \cdots 3N + 1$ if $d = 1$, where $f^{(k)}$ is the $k-$th derivative of f, and N is the same as in Condition (FGR),

(VB) V decays exponentially fast at ∞.

Theorem 5.1. *Let Conditions (fA)-(fC), (VA), (VB), (SA), (SB) and (FGR) be satisfied and let, for d ≥ 3, the potential V be spherically symmetric. Let an initial condition ψ_0 be permutation symmetric if $d \geq 3$ and $\lambda \in \mathcal{I}$. There exists $c > 0$ such that, if*

$$\inf_{\gamma\in\mathbb{R}}\{\|\psi_0-e^{i\gamma}(\phi^\lambda+z_1^{(0)}\xi+iz_2^0\eta)\|_{\mathcal{H}^k}+\|(1+x^2)^\nu[\psi_0-e^{i\gamma}(\phi^\lambda+z_1^{(0)}\xi+iz_2^0\eta)]\|_2\} \leq c[(z_1^0)^2+(z_2^0)^2] \tag{19}$$

with small real constants $z_1^{(0)}$ and $z_2^{(0)}$, some large constant $\nu > 0$ and with $k = [\frac{d}{2}] + 3$ if $d \geq 3$, and $k = 1$ if $d = 1$, then there exist differentiable functions $\gamma,\ z_1,\ z_2 : \mathbb{R}^+ \to \mathbb{R}$, $\lambda : \mathbb{R}^+ \to \mathcal{I}$ and $R : \mathbb{R}^+ \to \mathcal{H}^k$ such that the solution, $\psi(t)$, to Equation (1) is of the form

$$\psi(t) = e^{i\int_0^t \lambda(s)ds+i\gamma(t)}[\phi^{\lambda(t)} + z_1(t)\xi + iz_2(t)\eta + R(t)] \tag{20}$$

with the following estimates:

(A) $\|(1 + x^2)^{-\nu}R(t)\|_2 \leq c(1 + |t|)^{-\frac{1}{N}}$ *where ν is the same as in (19) and N is the same as that in (FGR),*

(B) $\sum_{j=1}^{2} |z_j(t)| \leq c(1 + t)^{-\frac{1}{2N}}$.

Remark 4. *Recall from Remark 3 that the class of permutationally symmetric data includes wave packets with initial momenta directed toward or in the opposite direction of the origin.*

Theorem 5.2. *Under the conditions of Theorem 3 we have*

(A) there exists a constant $\lambda_\infty \in \mathcal{I}$ such that $\lim_{t\to\infty} \lambda(t) = \lambda_\infty$.

(B) Let $z := z_1 - iz_2$. Then there exists a change of variables $\beta = z + O(|z|^2)$ such that

$$\dot{\beta} = i\epsilon(\lambda)\beta + \sum_{n=1}^{N} Y_n(\lambda)\beta^{n+1}\bar{\beta}^n + O(|\beta|^{2N+2}) \tag{21}$$

with Y_n being purely imaginary if $n < N$ and, by Condition (FGR) $ReY_N < 0$. Moreover, for $N = 1, 2, 3$, ReY_N is given by Equation (18). (Recall that $\epsilon(\lambda) = h\sqrt{\frac{2\Delta V(0)}{d}} + o(h)$.)

Remark 5. *Equations (20) and (21) can be rewritten in the form (3) with $a(t)$ and $p(t)$ satisfying the equations $\frac{1}{2}\dot{a} = p$ and $\dot{p} = -h^2\nabla V(a)$ modulo $O(|a|^2 + |p|^2)$.*

The proof of Theorems 5.1 and 5.2 are given in Sections 6- 10 for $d \geq 3$ and in Section 11 for $d = 1$. In order not to clutter the notation we restrict the arguments in Section 10 to the case $d = 3$ only.

6 Re-parametrization of $\psi(t)$

In this section we introduce a convenient decomposition of the solution $\psi(t)$ to Equation (1) into a solitonic component and a simplectically orthogonal fluctuation.

Theorem 6.1. *There exists a constant $\delta > 0$ such that if an initial condition $\psi(0)$ satisfies $\inf_{\gamma\in[0,2\pi)} \|\psi(0) - e^{i\gamma}\phi^\lambda\|_{\mathcal{H}^1} < \delta$, then for any time t $\psi(t)$ can be decomposed uniquely as*

$$\psi(t) = e^{i\int_0^t \lambda(s)ds + i\gamma(t)}(\phi^\lambda + z_1(t)\xi + iz_2(t)\eta + R(t)), \tag{22}$$

where $\lambda,\ \gamma,\ z_1,\ z_2$ are real differentiable functions of t, and the remainder $R(t)$ satisfies the orthogonality conditions

$$Im\langle R, i\phi^\lambda\rangle = Im\langle R, \frac{d}{d\lambda}\phi^\lambda\rangle = Im\langle R, i\eta\rangle = Im\langle R, \xi\rangle = 0. \tag{23}$$

Proof. By the Lyapunov stability (see [GSS1]), $\forall\ \epsilon > 0$, there exists a constant δ, such that if $\inf_{\gamma\in R} \|\psi(0) - e^{i\gamma}\phi^\lambda\|_{\mathcal{H}^1} < \delta$, then $\forall\ t > 0$, $\inf_{\gamma} \|\psi(t) - e^{i\gamma}\phi^\lambda\|_{\mathcal{H}^1} < \epsilon$. Then Decomposition (22) (23) follows from Splitting Theorem in [FGJS]. $\square$

After plugging Equation (22) into Equation (1), we get the equation

$$\begin{aligned} iR_t &= \mathcal{L}(\lambda)R + N(R, z_1, z_2) + \epsilon(\lambda)[iz_2\xi + z_1\eta] + \dot{\gamma}[\phi^\lambda + z_1\xi + iz_2\eta + R] \\ &\quad -i\dot{\lambda}\phi^\lambda_\lambda - i\dot{z}_1\xi - i\dot{\lambda}z_1\partial_\lambda\xi + \dot{z}_2\eta + \dot{\lambda}z_2\partial_\lambda\eta. \end{aligned} \tag{24}$$

where $\mathcal{L}(\lambda)$ is a real-linear operator given by

$$\mathcal{L}(\lambda)R := -\Delta R + \lambda R + V_h R - f((\phi^\lambda)^2)R - 2f'((\phi^\lambda)^2)(\phi^\lambda)^2 ReR,$$

and $N(R, z_1, z_2)$ is the nonlinear term given by

$$\begin{aligned} N(R, z_1, z_2) &:= -f(|\phi^\lambda + z_1\xi + iz_2\eta + R|^2)(\phi^\lambda + z_1\xi + iz_2\eta + R) \\ &\quad +f((\phi^\lambda)^2)(\phi^\lambda + z_1\xi + iz_2\eta + R) + 2f'((\phi^\lambda)^2)(\phi^\lambda)^2[z_1\xi + ReR]. \end{aligned} \tag{25}$$

Projecting Equation (24) onto the vectors $\phi^\lambda,\ \phi^\lambda_\lambda,\ \eta$ and ξ we derive the following equations for $\lambda,\ \gamma,\ z_1$ and z_2 as follows

$$\begin{array}{l}\dot\lambda[\delta'(\lambda) - Re\langle R, \phi^\lambda_\lambda\rangle] - \dot\gamma Im\langle R, \phi^\lambda\rangle = Im\langle N(R,z), \phi^\lambda\rangle,\\ \dot\gamma[\delta'(\lambda) + Re\langle R, \phi^\lambda_\lambda\rangle] - \dot\lambda Im\langle R, \phi^\lambda_{\lambda\lambda}\rangle = -Re\langle N(R,z), \phi^\lambda_\lambda\rangle,\end{array} \tag{26}$$

and

$$\begin{array}{rl} & [\dot z_1 - \epsilon(\lambda)z_2]\langle\xi,\eta\rangle \\ = & \dot\lambda Re\langle R, \eta_\lambda\rangle + Im\langle N(\vec R, z), \eta\rangle + \dot\gamma z_2\langle\eta,\eta\rangle + \dot\gamma Im\langle R,\eta\rangle - \dot\lambda z_1\langle\xi_\lambda,\eta\rangle; \\ & [\dot z_2 + \epsilon(\lambda)z_1]\langle\xi,\eta\rangle \\ = & \dot\lambda Im\langle R, \xi_\lambda\rangle - Re\langle N(\vec R, z), \xi\rangle - \dot\gamma z_1\langle\xi,\xi\rangle - \dot\gamma Re\langle R,\xi\rangle - \dot\lambda z_2\langle\eta_\lambda,\xi\rangle. \end{array} \tag{27}$$

As was already discussed above since the operator $\mathcal{L}(\lambda)$ is only real-linear we pass from the unknown R to the unknown $\vec R := \begin{pmatrix} ReR \\ ImR \end{pmatrix} \leftrightarrow R$. Under this correspondence the multiplication by i^{-1} goes over to the symplectic matrix

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}:\ J\vec R \leftrightarrow i^{-1}R.$$

Observe that due to (23)

$$\vec R \perp J\begin{pmatrix} 0 \\ \phi^\lambda \end{pmatrix},\ J\begin{pmatrix} \phi^\lambda_\lambda \\ 0 \end{pmatrix},\ J\begin{pmatrix} \xi \\ 0 \end{pmatrix},\ J\begin{pmatrix} 0 \\ \eta \end{pmatrix}. \tag{28}$$

On the other hand in Equations (27) it is more convenient to go from the real, symplectic structure given by J to the complex structure i^{-1} by passing from $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ to $z := z_1 - iz_2$. Let $\vec N(\vec R, z) := \begin{pmatrix} ReN(R, z_1, z_2) \\ ImN(R, z_1, z_2) \end{pmatrix}$. Then

$$\frac{d}{dt}\vec R = L(\lambda)\vec R + \dot\gamma J\vec R + J\vec N(\vec R, z) + \begin{pmatrix} z_2\epsilon(\lambda)\xi + \dot\gamma z_2\eta - \dot\lambda\phi^\lambda_\lambda - \dot z_1\xi - \dot\lambda z_1\xi_\lambda \\ -z_1\epsilon(\lambda)\eta - \dot\gamma\phi^\lambda - \dot\gamma z_1\xi - \dot z_2\eta - \dot\lambda z_2\eta_\lambda \end{pmatrix} \tag{29}$$

where $z_1 = \mathrm{Re}z, z_2 = \mathrm{Im}z$ and the linear operator $L(\lambda)$ is given by (10)-(12).

Define P_d as the Riez projection for the isolated eigenvalues of $L(\lambda)$. It was shown in [GS1] that (in the Dirac notation)

$$\begin{array}{rcl} P_d & = & \frac{1}{\delta'(\lambda)}\left(\left| \begin{array}{l} 0 \\ \phi^\lambda \end{array} \right\rangle \left\langle \begin{array}{l} \frac{d}{d\lambda}\phi^\lambda \\ 0 \end{array} \right| + \left| \begin{array}{l} \frac{d}{d\lambda}\phi^\lambda \\ 0 \end{array} \right\rangle \left\langle \begin{array}{l} 0 \\ \phi^\lambda \end{array} \right|\right) \\ & & + \frac{i}{2\langle\xi,\eta\rangle}\left(\left| \begin{array}{l} \xi \\ i\eta \end{array} \right\rangle \left\langle \begin{array}{l} -i\eta \\ \xi \end{array} \right| + \left| \begin{array}{l} -\xi \\ i\eta \end{array} \right\rangle \left\langle \begin{array}{l} i\eta \\ \xi \end{array} \right|\right). \end{array} \tag{30}$$

We denote $P_c := 1 - P_d$. We call P_c the projection onto the essential spectrum of $L(\lambda)$.

Since $P_c\vec R = \vec R$, we have that

$$P_c\frac{d}{dt}\vec R = \frac{d}{dt}\vec R - \dot\lambda P_{c\lambda}\vec R.$$

Applying the projection P_c to Equation (29) and using the relations above we find

$$\begin{aligned} \frac{d}{dt}\vec{R} &= L(\lambda)\vec{R} + \dot{\lambda} P_{c\lambda}\vec{R} + \dot{\gamma} P_c J\vec{R} + P_c J\vec{N}(\vec{R}, z) \\ &+ \tfrac{1}{2}\dot{\gamma} P_c[z\begin{pmatrix} -i\eta \\ \xi \end{pmatrix} + \bar{z}\begin{pmatrix} i\eta \\ \xi \end{pmatrix}] - \tfrac{1}{2}\dot{\lambda} P_c[z\begin{pmatrix} \xi_\lambda \\ -i\eta_\lambda \end{pmatrix} + \bar{z}\begin{pmatrix} \xi_\lambda \\ i\eta_\lambda \end{pmatrix}]. \end{aligned} \tag{31}$$

On the other hand Equations (27) for z_1 and z_2 become

$$\begin{aligned} \dot{z} &= i\epsilon(\lambda)z + \tfrac{1}{\langle \xi, \eta\rangle}\langle J\vec{N}(\vec{R}, z) + \dot{\gamma}\begin{pmatrix} z_2\eta \\ z_1\xi \end{pmatrix} + \dot{\gamma} J\vec{R} - \dot{\lambda}\begin{pmatrix} z_1\xi_\lambda \\ -z_2\eta_\lambda \end{pmatrix}, \begin{pmatrix} \eta \\ -i\xi \end{pmatrix}\rangle \\ &+ \tfrac{\dot{\lambda}}{\langle \xi, \eta\rangle}\langle \vec{R}, \begin{pmatrix} \eta_\lambda \\ -i\xi_\lambda \end{pmatrix}\rangle. \end{aligned} \tag{32}$$

Finally, Equation (26) for λ and γ can be rewritten as

$$\begin{pmatrix} \delta'(\lambda) + \langle R_1, \phi^\lambda_\lambda\rangle & -\langle R_2, \phi^\lambda_{\lambda\lambda}\rangle \\ -\langle R_2, \phi^\lambda\rangle & \delta'(\lambda) - \langle R_1, \phi^\lambda_{\lambda\lambda}\rangle \end{pmatrix}\begin{pmatrix} \dot{\gamma} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} -Re\langle N(\vec{R}, z), \phi^\lambda_\lambda\rangle \\ Im\langle N(\vec{R}, z), \phi^\lambda\rangle \end{pmatrix}. \tag{33}$$

Remark 6. *By the gauge invariance of Equation (1), Equations (31)-(33) are invariant under the gauge transformation, $\gamma \to \gamma + \alpha$, for any $\alpha \in \mathbb{R}$, and other parameters fixed. Hence these equations and their solutions are independent of γ.*

7 Expansions of the Functions $\vec{R}$, $\dot{\lambda}$ and $\dot{\gamma}$

In this section we construct expansions of the functions $\vec{R}$, $\dot{\lambda}$, $\dot{z}$ and $\dot{\gamma}$ in the parameter

$$z := z_1 - iz_2.$$

In what follows we fix N to be the smallest positive integer such that $(N+1)\epsilon(\lambda) > \lambda$, where, recall, that $i\epsilon(\lambda)$ and $-i\epsilon(\lambda)$ are the only nonzero eigenvalues of $L(\lambda)$.

Definition 7.1. *A vector-function $\vec{u} : \mathbb{R}^d \to \mathbb{C}^2$ is admissible if the vector-function $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\vec{u}$ has real entries.*

Theorem 7.2. *There exists a polynomial $P(z, \bar{z}) = \sum_{2\le m+n\le N} a_{m,n}(\lambda)z^m\bar{z}^n$ with $a_{m,n}(\lambda) \in \mathbb{R}$ such that if we define $y := z + P(z, \bar{z})$ then for any $2 \le k \le 2N$, the function $\vec{R}$ can be decomposed as*

$$\vec{R} = \sum_{2\le m+n\le k} R_{mn}(\lambda)y^m\bar{y}^n + R_k \tag{34}$$

where the functions $R_{mn}(\lambda)$, $R_k : \mathbb{R}^3 \to \mathbb{C}^2$ have the following properties:

(RA) if $\max\{m,n\} \le N$, *then the functions* $R_{m,n}(\lambda) \in \mathcal{L}^2$ *are admissible, and decay exponentially fast at* ∞;

(RB) if $\max\{m,n\} > N$ *then the functions* $R_{m,n}(\lambda)$ *are of the form*

$$\prod_k (L(\lambda) - ik\epsilon(\lambda) + 0)^{-n_k} P_c \phi_{m,n}(\lambda), \tag{35}$$

where the functions $\phi_{m,n}(\lambda)$ *are smooth and decay exponentially fast at* ∞, $0 \le \sum_k n_k \le N$, *and* $2N \ge k \ge N+1$; *note that the equation (35) makes sense in an appropriate weighted* $\mathcal{L}^2$ *space (see Section 10.1);*

(RC) the function R_k $(N \le k \le 2N)$ *satisfies the equation*

$$\frac{d}{dt} R_k = L(\lambda) R_k + P_k(y,\bar{y}) R_k + N_N(R_N, y, \bar{y}) + F_k(y,\bar{y}), \tag{36}$$

where

(1) $F_k(y,\bar{y}) = O(|y|^{k+1})$ *is a polynomial in* y *and* $\bar{y}$ *with* λ*-function-valued coefficients, and each coefficient can be written as the sum of functions of the form (35);*

(2) $P_k(y,\bar{y})$ *is the operator defined by*

$$P_k(y,\bar{y}) := \dot{\gamma} P_c J + \dot{\lambda} P_{c\lambda} + A_k(y,\bar{y}),$$

where $A_k(y,\bar{y})$ *is a* 2×2 *matrix-valued function of* y, $\bar{y}$, x *and* λ *bounded in the matrix norm as*

$$|A_k(y,\bar{y})| \le c|y| e^{-\epsilon_0 |x|};$$

(3) $N_N(R_N, y, \bar{y})$ *satisfies the estimates*

$$\|N_N(R_N,y,\bar{y})\|_1 + \|N_N(R_N,y,\bar{y})\|_{\mathcal{H}^l} \le c(1+t)^{-\frac{2N+3}{2N}} [Y^2 \mathcal{R}_1^6 + \mathcal{R}_1^5 \mathcal{R}_2^2], \tag{37}$$

$$\begin{aligned} &\|(-\Delta+1)^{\frac{l}{2}} N_N(R_N,y,\bar{y})\|_1 + \|N_N(R_N,y,\bar{y})\|_{\mathcal{H}^l} \\ \le\ & c(T_0+t)^{-\frac{2N+3}{2N}} (Y^2 \mathcal{R}_1^2 \mathcal{R}_2^2 + Y^2 \mathcal{R}_1^3 \mathcal{R}_2^3 + \mathcal{R}_1^3 \mathcal{R}_2^4), \end{aligned} \tag{38}$$

where the constant l *is defined as* $l := [\frac{d}{2}] + 3$, *the estimating functions* Y, $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *depend on* t *and are defined in Equations (41) below.*

Furthermore, for z *satisfying Equation (32), the parameter* y *satisfies the equation*

$$\dot{y} = i\epsilon(\lambda) y + \sum_{2 \le m+n \le 2N+1} \Theta_{mn}(\lambda) y^m \bar{y}^n + Remainder, \tag{39}$$

where $\Theta_{mn}(\lambda)$ *is purely imaginary for* $m,n \le N$; $\Theta_{m,n}(\lambda) = 0$ *for* $m+n \le N$ *and* $m \ne n+1$. *The term Remainder is bounded as*

$$|Remainder(t)| \le c(T_0+t)^{-\frac{2N+2}{2N}} (Y(t) + \mathcal{R}_1(t))^2. \tag{40}$$

Above we used the following functions:

$$
\begin{aligned}
&Y(T) := \max_{t\leq T}(T_0+t)^{\frac{1}{2N}}|y(t)|,\\
&\mathcal{R}_1(T) := \max_{t\leq T}[(T_0+t)^{\frac{N+1}{2N}}(\|\rho_\nu R_N\|_{\mathcal{H}^l}+\|R_N(t)\|_\infty)+(T_0+t)^{\frac{2N+1}{2N}}\|\rho_\nu R_{2N}(t)\|_2]\\
&\mathcal{R}_2(T) := \max_{t\leq T}\|R_N(t)\|_{\mathcal{H}^l}
\end{aligned}
\tag{41}
$$

where $l := [\frac{d}{2}]+3$, $T_0 := (|z_1^{(0)}|+|z_2^{(0)}|)^{-1}$, ν is some large defined in (82) below, recall the definitions of $z_1^{(0)}$ and $z_2^{(0)}$ in Main Theorem 5.1.

Our next result is

Theorem 7.3. *The functions $\dot{\lambda}$ and $\dot{\gamma}$ have the following expansions in the parameters y and $\bar{y}$:*

$$\dot{\lambda} = \sum_{2\leq m+n\leq 2N+1} \Lambda_{mn}(\lambda)y^m\bar{y}^n + Remainder, \tag{42}$$

where $\Lambda_{mn}(\lambda) = \bar{\Lambda}_{nm}(\lambda)$, and $\Lambda_{m,n}(\lambda)$ is purely imaginary for $m, n \leq N$, (therefore $\Lambda_{mm}(\lambda) = 0$ for $m \leq N$);

$$\dot{\gamma} = \sum_{2\leq m+n\leq 2N+1} \Gamma_{mn}(\lambda)y^m\bar{y}^n + Remainder, \tag{43}$$

where $\Gamma_{mn}(\lambda) = \bar{\Gamma}_{nm}(\lambda)$, and $\Gamma_{mn}(\lambda)$ is real for $m, n \leq N$. The terms Remainder are not the same in the equations above, but both admit the estimate (40).

7.1 Proof of Theorems 7.2 and 7.3

In this subsection we prove Theorems 7.2 and 7.3. We divide the proof into three steps. The following lemma will be used repeatedly to prove the admissibility of the function $R_{m,n}(\lambda)$.

Lemma 7.4. *If K_1 is a vector-function from $\mathbb{R}^d$ to $\mathbb{C}^2$ such that iK_1 is admissible, then the vector function*

$$K_2 := (L(\lambda)-i\mu)^{-1}P_cK_1$$

is admissible for any $\mu \in (-\lambda, \lambda)$.

Proof. First by Equation (30) we observe that iP_cK_1 is admissible. Then the computation

$$
\begin{aligned}
\bar{K}_2 &= (L(\lambda)+i\mu)^{-1}P_c\bar{K}_1\\
&= -(L(\lambda)+i\mu)^{-1}\sigma_3P_cK_1\\
&= \sigma_3(L(\lambda)-i\mu)^{-1}P_cK_1\\
&= \sigma_3K_2,
\end{aligned}
$$

where $\sigma_3 := \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}$, implies that K_2 is admissible. ∎

7.1.1 The first step: z-expansion

In this sub-subsection we prove the following proposition.

Proposition 7.5. *For any $k = 2, \cdots, N$*

$$\vec{R} = \sum_{2 \le m+n \le k} \tilde{R}_{m,n}(\lambda) z^m \bar{z}^n + \tilde{R}_k \tag{44}$$

where the functions $\tilde{R}_{m,n}$ are admissible, and the remainder $\tilde{R}_k$ satisfies the equation

$$\begin{aligned} \frac{d}{dt}\tilde{R}_k &= L(\lambda)\tilde{R}_k + \dot{\gamma} P_c J \tilde{R}_k + \dot{\lambda} P_{c\lambda} \tilde{R}_k + \tilde{A}_k(z,\bar{z})\tilde{R}_k \\ &+ \sum_{k+1 \le m+n \le 2N} R^{(1)}_{m,n} z^m \bar{z}^n + N_k(\tilde{R}_k, z) + Remainder_1 \end{aligned} \tag{45}$$

where the term $N_k(\tilde{R}_k, z)$ contains all the nonlinear terms in $\tilde{R}_k$ and, for $k = N$, is bounded as

$$\|N_N(\tilde{R}_N, z)\|_1 + \|N_N(\tilde{R}_N, z)\|_{\mathcal{H}^l} \le c(T_0 + t)^{-\frac{2N+3}{2N}} [Y^2 \mathcal{R}_1^6 + \mathcal{R}_1^5 \mathcal{R}_2^2],$$

$$\begin{aligned} &\|(-\Delta + 1)^{\frac{l}{2}} N_N(\tilde{R}_N, z)\|_1 + \|N_N(\tilde{R}_N, z)\|_{\mathcal{H}^l} \\ \le\ & c(T_0 + t)^{-\frac{2N+2}{2N}} (Y^2 \mathcal{R}_1^2 \mathcal{R}_2^2 + Y^2 \mathcal{R}_1^3 \mathcal{R}_2^3 + \mathcal{R}_1^3 \mathcal{R}_2^4) \end{aligned}$$

with $l := [\frac{d}{2}] + 3$, the functions $iR^{(1)}_{m,n}$ are admissible, smooth, and decay exponentially fast at ∞, and the (2×2)-matrix function $\tilde{A}_k(z, \bar{z})$ is bounded in the matrix norm as

$$|\tilde{A}_k(z, \bar{z})| \le c|z||e^{-\epsilon_0 |x|},$$

and the function $Remainder_1$ satisfies the estimate

$$|Remainder_1| \le c|z|^{2N+1} e^{-\epsilon_0 |x|}. \tag{46}$$

Proof. We prove the theorem by induction in k. Thus we first consider the case $k = 2$. If we let

$$\tilde{R}_{2,0}(\lambda) := \frac{1}{4}[L(\lambda) - 2i\epsilon(\lambda)]^{-1} P_c f^{'}((\phi^\lambda)^2)\phi^\lambda \begin{pmatrix} 2i\eta\xi \\ 3\xi^2 + \eta^2 \end{pmatrix},$$

$$\tilde{R}_{0,2}(\lambda) := \bar{\tilde{R}}_{2,0}(\lambda),$$

$$\tilde{R}_{11}(\lambda) := \frac{1}{2} L(\lambda)^{-1} P_c f^{'}((\phi^\lambda)^2)\phi^\lambda \begin{pmatrix} 0 \\ 3\xi^2 - \eta^2 \end{pmatrix}$$

and

$$\tilde{R}_2 := \vec{R} - \sum_{m+n=2} z^m \bar{z}^n \tilde{R}_{m,n}(\lambda),$$

then the functions $\tilde{R}_{m,n}(\lambda)$, $m + n = 2$, are admissible by Lemma 7.4 and $\tilde{R}_2$ satisfies the equation (45) when $k = 2$. Thus we obtain the first step of induction.

Now assume (44) holds for some $2 \le k-1 < N$ and prove it for k. Define the term $\tilde{R}_k$ by (44). We claim that $\tilde{R}_k$ satisfies the following equation:

$$\begin{aligned} \tfrac{d}{dt}\tilde{R}_k &= [L(\lambda)+\dot{\gamma}P_cJ+\dot{\lambda}P_{c\lambda}+\tilde{A}_k(z,\bar{z})]\tilde{R}_k \\ &\quad + \sum_{2\le m+n\le 2N} z^m\bar{z}^n F_{m,n} + P_cJN_k(\tilde{R}_k,z) + Remainder_1 \end{aligned} \tag{47}$$

where for $m+n \le k$

$$F_{m,n} := [L(\lambda) - i\epsilon(\lambda)(m-n)]\tilde{R}_{mn}(\lambda) + P_c f_{m,n}(\lambda),$$

with functions $f_{m,n}(\lambda)$ having the following properties

(A) the functions $f_{m,n}(\lambda)$ depend on $\tilde{R}_{m',n'}(\lambda)$ with $m'+n' < m+n$;

(B) $if_{m,n}(\lambda)$ are admissible, smooth and decays exponentially fas,t provided that $\tilde{R}_{m',n'}(\lambda)$ are admissible, smooth and decay exponentially fast for all pairs (m',n') satisfying $m'+n' < m+n$.

We prove this claim below. Recall that if $|m-n| \le N$, then $i\epsilon(\lambda)(m-n) \notin \sigma(L(\lambda))$ and therefore the operators

$$L(\lambda) - i\epsilon(\lambda)(m-n) : P_c\mathcal{L}^2 \to P_c\mathcal{L}^2 \tag{48}$$

are invertible. Hence by Lemma 7.4 the equations $F_{m,n}(\lambda) = 0$, $m+n \le k \le N$, have unique solutions with the property that $\tilde{R}_{m,n}(\lambda)$ are admissible if $if_{m,n}(\lambda)$ are admissible. By Claim (B), $if_{m,n}(\lambda)$ are admissible if $\tilde{R}_{m',n'}(\lambda)$, $m'+n' < m+n$, are admissible. This and the induction in k show the admissibility of $\tilde{R}_{m,n}(\lambda)$ for $m+n \le N$.

What is left is to prove the claims above. To this latter end we plug decomposition (44) into Equation (31) to obtain

$$\begin{aligned} \tfrac{d}{dt}\tilde{R}_k &= L(\lambda)\tilde{R}_k + F(\vec{R},z) + \sum_{2\le m+n\le k} z^m\bar{z}^n[L(\lambda) - i(m-n)\epsilon(\lambda)]\tilde{R}_{m,n}(\lambda) \\ &\quad - \sum_{2\le m+n\le k} \tilde{R}_{m,n}(\lambda)[\frac{d}{dt}z^m\bar{z}^n - i(m-n)\epsilon(\lambda)z^m\bar{z}^n] \\ &\quad -\dot{\lambda}\sum_{2\le m+n\le k} \partial_\lambda\tilde{R}_{m,n}(\lambda)z^m\bar{z}^n \end{aligned}$$

where the term $F(\vec{R},z)$ is given by

$$\begin{aligned} F(\vec{R},z) &:= \dot{\lambda}P_{c\lambda}\vec{R} + \dot{\gamma}P_cJ\vec{R} + P_cJ\vec{N}(\vec{R},z) \\ &\quad + \tfrac{1}{2}\dot{\gamma}P_c[z\begin{pmatrix} -i\eta \\ \xi \end{pmatrix} + \bar{z}\begin{pmatrix} i\eta \\ \xi \end{pmatrix}] - \tfrac{1}{2}\dot{\lambda}P_c[z\begin{pmatrix} \xi_\lambda \\ -i\eta_\lambda \end{pmatrix} + \bar{z}\begin{pmatrix} \xi_\lambda \\ i\eta_\lambda \end{pmatrix}]. \end{aligned}$$

Moreover $J\vec{N}(\vec{R},z) := J\begin{pmatrix} ReN(R,z_1,z_2) \\ ImN(R,z_1,z_2) \end{pmatrix}$ admits the expansion

$$J\vec{N}(\vec{R},z) = \sum_{2\le m+n\le 2N} z^m\bar{z}^n N_{mn}(\lambda) + \tilde{A}_k(z,\bar{z})\tilde{R}_k + N_k(\tilde{R}_k,z) + Remainder_1 \tag{49}$$

where $N_{m,n}(\lambda) : \mathbb{R}^3 \to \mathbb{C}^2$, $\tilde{A}_k(z,\bar{z})$ as above, $N_k(\tilde{R}_k, z)$ contains all the nonlinear terms in $\tilde{R}_k$ and the term $Remainder_1$ has the same estimate as in Equation (46).

By Equations (25), (32) and (33) for $\vec{N}(\vec{R}, z)$, $\dot{z}$, $\dot{\lambda}$ and $\dot{\gamma}$, to prove the claim it suffices to prove that given (m,n), the function $iN_{m,n}(\lambda)$ in Equation (49) is admissible if $\tilde{R}_{m',n'}(\lambda)$ are admissible for all $m'+n' < m+n$, and depends only on $\tilde{R}_{m',n'}(\lambda)$, $m'+n' < m+n$. The proof of this sufficient condition is tedious and not hard, thus omitted.

(1) To prove the admissibility of $iN_{m,n}(\lambda)$ we use the definition of $N(R, z_1, z_2)$ in Equation (25) again. Note that if $f_{m,n}(\lambda)$ and $F_{m',n'}(\lambda)$ are real and admissible functions, respectively, then the vector-function $f_{m,n}F_{m',n'}(\lambda)$ is admissible. Therefore it is sufficient to prove that if $\tilde{R}_{m',n'}(\lambda)$, $m'+n' < m+n$, are admissible, then we have the expansion

$$\begin{aligned} f(|\phi^\lambda + z_1\xi + iz_2\eta + R|^2) &= f((\phi^\lambda)^2) + \sum_{1\leq m+n\leq 2N} f_{m,n}(\lambda) z^m \bar{z}^n + g(\tilde{R}_k) \\ &\quad + Remainder_1 \end{aligned} \tag{50}$$

where the functions $f_{m,n}(\lambda)$ are real, g collects all the linear and nonlinear terms containing $\tilde{R}_k$; it obeys the estimate

$$\|g(\tilde{R}_k)\|_2 \leq c(|z|^4 \|e^{-\epsilon_0|x|}\tilde{R}_k\|_2 + \|\tilde{R}_k^3\|_2)$$

for some constant $\epsilon_0 > 0$, and $Remainder_1$ satisfies the estimate (46). Indeed, let $\vec{\phi} := \begin{pmatrix} \phi^\lambda + \frac{z+\bar{z}}{2}\xi \\ \frac{z-\bar{z}}{2i}\eta \end{pmatrix}$, then

$$|\phi^\lambda + z_1\xi + iz_2\eta + R|^2 = |\vec{\phi} + \vec{R}|^2,$$

where, recall that $\vec{R} := \begin{pmatrix} ReR \\ ImR \end{pmatrix}$. Let $\vec{R}_k := \sum_{2\leq m+n\leq k} \tilde{R}_{m,n}(\lambda) z^m \bar{z}^n$ be a real function in Equation (44). We find

$$|\vec{\phi} + \vec{R}|^2 = \vec{\phi}\cdot\vec{\phi} + 2\vec{\phi}\cdot\vec{R}_k + \vec{R}_k\cdot\vec{R}_k + 2(\vec{\phi} + \vec{R}_k)\cdot\tilde{R}_k + \tilde{R}_k\cdot\tilde{R}_k.$$

Since the vector-functions $\vec{\phi}$ and $\vec{R}_k$ have finite $z-$expansions with admissible coefficients, the first three functions on the right hand side have finite $z-$expansions with real coefficients. Moreover the expansion for $\vec{\phi}\cdot\vec{\phi}$ starting with the term $(\phi^\lambda)^2$. Expanding the function $f(|\phi^\lambda + z_1\xi + iz_2\eta + R|^2)$ around $(\phi^\lambda)^2$ to the $2N-$th order, we have Equation (50).

(2) The fact that $N_{mn}(\lambda)$ depends only on the terms $\tilde{R}_{m',n'}$, $m'+n' < m+n$, follows from by the computation in Statement (1) above.

□

We plug the expansion of the function $\vec{R}$ into (32) and (33) to obtain the following expansions for $\dot{\lambda},\ \dot{\gamma}$ and $\dot{z}$:

Corollary 7.6.

$$\begin{aligned}\dot{\gamma} &= \sum_{2\leq m+n\leq 2N+1} \tilde{\Gamma}_{m,n}(\lambda) z^m \bar{z}^n + \tilde{l}_\gamma(\tilde{R}_N) + Remainder;\\ \dot{\lambda} &= \sum_{2\leq m+n\leq 2N+1} \tilde{\Lambda}_{m,n}(\lambda) z^m \bar{z}^n + \tilde{l}_\lambda(\tilde{R}_N) + Remainder;\end{aligned} \tag{51}$$

$$\dot{z} = i\epsilon(\lambda) z + \sum_{2\leq m+n\leq 2N+1} \tilde{Z}_{m,n}(\lambda) z^m \bar{z}^n + \tilde{l}_z(\tilde{R}_N) + Remainder \tag{52}$$

where, $\tilde{l}_\gamma,\ \tilde{l}_\lambda$ and $\tilde{l}_z$ are linear functionals having the estimates

$$|\tilde{l}_\gamma(g)|,\ |\tilde{l}_\lambda(g)|,\ |\tilde{l}_z(g)| \leq c|z| \|e^{-\epsilon_0|x|} g\|_2, \tag{53}$$

the term Remainder admits the same estimate as in (40), the coefficients $\tilde{\Gamma}_{m,n}(\lambda)$ are real, and $\tilde{\Lambda}_{m,n}(\lambda)$ and $\tilde{Z}_{m,n}(\lambda)$ are purely imaginary.

The proof is straightforward by Proposition 7.5, Equations (33) (32) and the properties of the term $J\vec{N}(\vec{R}, z)$ in (49), and thus is omitted.

7.1.2 The second step: changing variables

In the second step we transform z to a parameter y which satisfies a simpler different equation.

Proposition 7.7. *There exists a polynomial $P(z, \bar{z})$ with real coefficients and the smallest degree ≥ 2, such that if we define $y := z + P(z, \bar{z})$ then*

$$\dot{y} = i\epsilon(\lambda) y + \sum_{2\leq m+n\leq 2N+1} Y_{m,n}(\lambda) y^m \bar{y}^n + l_y(\tilde{R}_N) + Remainder \tag{54}$$

where the coefficients $Y_{m,n}(\lambda)$ are purely imaginary, especially $Y_{m,n} = 0$ if $m + n \leq N$ and $m \neq n+1$, l_y is a linear functional satisfying the estimate

$$|l_y(g)| \leq c|y| \|e^{-\epsilon_0|x|} g\|_2,$$

and the term Remainder admits the estimate (40).

Proof. We show how to construct the polynomial $P(z, \bar{z})$. We rewrite Equation (52) as

$$\begin{aligned}&\partial_t\Big(z - \sum_{m+n=2} \frac{\tilde{Z}_{m,n}(\lambda)}{i(m-n-1)\epsilon(\lambda)} z^m \bar{z}^n\Big)\\ =\ & i\epsilon(\lambda)\Big[z - \sum_{m+n=2} \frac{\tilde{Z}_{m,n}(\lambda)}{i(m-n-1)\epsilon(\lambda)} z^m \bar{z}^n\Big] + D + \tilde{l}_z(\tilde{R}_N) + Remainder,\end{aligned} \tag{55}$$

where the linear functional $\tilde{l}_z$ satisfies the same estimate as in (53), the term D is given by

$$\begin{aligned} D \;:=\; & -\tfrac{d}{dt}\sum_{m+n=2}\frac{\tilde{Z}_{m,n}(\lambda)}{i(m-n-1)\epsilon(\lambda)}z^m\bar{z}^n+\sum_{m+n=2}\frac{\tilde{Z}_{m,n}(\lambda)}{m-n-1}z^m\bar{z}^n \\ & +\sum_{2\leq m+n\leq 2N+1}\tilde{Z}_{m,n}(\lambda)z^m\bar{z}^n. \end{aligned}$$

Take the time derivative on the right hand side and use Equations (51) and (52) to get

$$D=\sum_{3\leq m+n\leq 2N+1}a^{(1)}_{m,n}(\lambda)z^m\bar{z}^n+Remainder.$$

Since $\tilde{Z}_{m,n}(\lambda)$ and $\tilde{\Lambda}_{m,n}$ are purely imaginary we see that $a^{(1)}_{m,n}(\lambda)$ are purely imaginary. Now define

$$P_1(z,\bar{z}):=-\sum_{m+n=2}\frac{\tilde{Z}_{mn}(\lambda)}{i(m-n-1)\epsilon(\lambda)}z^m\bar{z}^n \tag{56}$$

and $y_1:=z+P_1(z,\bar{z})$. We observe that $\frac{\tilde{Z}_{m,n}(\lambda)}{i(m-n)\epsilon(\lambda)}$ are real. Then Equation (55) yields

$$\dot{y}_1=i\epsilon(\lambda)y_1+\sum_{3\leq m+n\leq 2N+1}a^{(2)}_{m,n}(\lambda)y_1^m\bar{y}_1^n+l_{y_1}(\tilde{R}_N)+Remainder$$

where $a^{(2)}_{m,n}(\lambda)$ are purely imaginary, and the term *Remainder* has the same estimate as in (40).

Next we remove the terms with $m+n=3$ and $m\neq n+1$ and so forth arriving at the end at Equation (54). □

We invert the relations $y=z+P(z,\bar{z})$ and $\bar{y}=\bar{z}+\bar{P}(z,\bar{z})$ and express the variables z and $\bar{z}$ as power series in y and $\bar{y}$. Plug the result into (51) for $\dot{\gamma}$ and $\dot{\lambda}$ and into Equations (44) for $\vec{R}$ to obtain the expansions

$$\dot{\gamma}=\sum_{2\leq m+n\leq 2N+1}\Gamma^{(1)}_{m,n}(\lambda)y^m\bar{y}^n+l_\gamma(R_N)+Remainder,$$

$$\dot{\lambda}=\sum_{2\leq m+n\leq 2N+1}\Lambda^{(1)}_{m,n}(\lambda)y^m\bar{y}^n+l_\lambda(R_N)+Remainder;$$

$$\vec{R}=\sum_{2\leq m+n\leq k}R_{m,n}(\lambda)y^m\bar{y}^n+R_k,$$

where $\Gamma^{(1)}_{m,n}(\lambda)$ are real, $\Lambda^{(1)}_{m,n}(\lambda)$ are purely imaginary, $2\leq k\leq N$, the linear functionals $l_\gamma,\ l_\lambda$ satisfy the estimate

$$|l_\gamma(g)|,|l_\lambda(g)|\leq c|y|\|e^{-\epsilon_0|x|}g\|_2,$$

$R_{m,n}$ are admissible, and R_k satisfies the equation

$$\begin{aligned}\tfrac{d}{dt}R_k &= L(\lambda)R_k + \dot{\gamma}P_cJR_k + \dot{\lambda}P_{c\lambda}R_k + A_k(y,\bar{y})R_k \\ &+ \sum_{N+1\le m+n\le 2N+1} iR^{(k)}_{m,n}(\lambda)y^m\bar{y}^n + N_k(R_k,y) + Remainder_1,\end{aligned} \tag{57}$$

with the functions $R^{(k)}_{m,n}(\lambda)$ admissible, and $N_N(R_N,y)$ satisfying the estimates (37)-(38) and the operator $A_k(y,\bar{y})$ have the same estimates as that in (45), and the terms *Remainder* and *Remainder*$_1$ admit the same estimates as in (40) and (46), respectively. Note that the polynomial P_1 in (56) has real coefficients and therefore the expansion of z and $\bar{z}$ in powers of y and $\bar{y}$ has real coefficients also. Since a product of real and admissible functions is admissible we conclude that the coefficients $R_{m,n}(\lambda)$ are also admissible.

The above relations prove Theorem 7.2, except for (39), for $2 \le k \le N$. The proof for $N < k \le 2N$ is more difficult since $i\epsilon(\lambda)(m-n)$ in (48) might be in the spectrum of $L(\lambda)$. This is done in the next step.

7.1.3 The Third Step: $N < k \le 2N$. Completion of the Proof of Theorems 7.2 and 7.3

This is the last and more involved step. As in the first step we determine the coefficients $R_{m,n}(\lambda)$ by solving the equations

$$[L(\lambda) - i\epsilon(\lambda)(m-n)]R_{m,n}(\lambda) = -P_cf_{m,n}(\lambda)$$

for certain functions $f_{m,n}(\lambda)$ (see below). Recall that the number N is defined by the properties

$$i\epsilon(\lambda)(m-n) \begin{array}{l}\notin \sigma(L(\lambda)) \text{ if } |m-n| \le N, \\ \in \sigma(L(\lambda)) \text{ if } |m-n| > N.\end{array}$$

Thus we sort out the pairs (m,n) into "non-resonant pairs" satisfying $|m-n| \le N$ and "resonant pairs" satisfying $|m-n| > N$. For "non-resonant" pairs the operators

$$L(\lambda) - i\epsilon(\lambda)(m-n) : P_c\mathcal{L}^2 \to P_c\mathcal{L}^2$$

are invertible and for resonant pairs they are not (one has to change spaces in the latter case). In the first two steps we expanded in z and $\bar{z}$ (and in y and $\bar{y}$) until $m+n \le N$ and consequently all the pairs, (m,n), involved were non-resonant ones. Now, for $k > N$, our expansion involves pairs (m,n) with $m+n > N$, which include resonant pairs. What we want to show now is that for the subsets of pairs (m,n), $m+n > N$, determined by the inequality

$$m, n \le N,$$

our analysis will involve only "non-resonant" pairs and we will be able to prove the admissibility of the coefficients $R_{m,n}(\lambda)$ in this case.

Definition 7.8. *Suppose that (m_1, n_1) and (m_2, n_2) are two pairs of nonnegative integers. Then $(m_1, n_1) < (m_2, n_2)$ if $m_1 \leq m_2$, $n_1 \leq n_2$ and $(m_1, n_1) \neq (m_2, n_2)$; and $(m_1, n_1) \leq (m_2, n_2)$ if $m_1 \leq m_2$, $n_1 \leq n_2$.*

To prove Theorem 7.2 for $N + 1 \leq k \leq 2N$ we proceed as in the proof of the first step.

Lemma 7.9. *Let $N < k \leq 2N$. Then the remainder term R_k in Equation (34) satisfies the equation*

$$\begin{aligned}\tfrac{d}{dt} R_k &= L(\lambda) R_k + P_k(y, \bar{y}) R_k + \sum_{2 \leq m+n \leq k} y^m \bar{y}^n F_{m,n}(\lambda) \\ &\quad + P_c J \vec{N}_N(\tilde{R}_N, y, \bar{y}) + F_k(y, \bar{y}),\end{aligned}$$

where $P_k(y, \bar{y})$, $N_N(R_N, y, \bar{y})$ and $F_k(y, \bar{y})$ are described in Theorem 7.2; $F_{mn}(\lambda)$ are the functions defined as

$$F_{m,n} := [L(\lambda) - i\epsilon(\lambda)(m - n)] R_{m,n}(\lambda) + P_c f_{m,n}(\lambda)$$

where the functions $f_{m,n}(\lambda)$ have the following properties:

(A) if $m, n \leq N$ and all the terms $R_{m_1,n_1}(\lambda)$, $(m_1, n_1) < (m, n)$, are admissible then $i f_{m,n}$ is admissible;

(B) if $\max\{m, n\} > N$ then $f_{m,n}(\lambda)$ is of the form (35).

Moreover we have the following expansions for $\dot{y}$, $\dot{\lambda}$ and $\dot{\gamma}$:

$$\dot{y} = i\epsilon(\lambda) y + \sum_{2 \leq m+n \leq 2N+1} \Theta_{m,n}(\lambda) y^m \bar{y}^n + l_y^{(k)}(R_k) + \mathit{Remainder}, \tag{58}$$

$$\dot{\lambda} = \sum_{2 \leq m+n \leq 2N+1} \Lambda_{m,n}(\lambda) y^m \bar{y}^n + l_\lambda^{(k)}(R_k) + \mathit{Remainder} \tag{59}$$

and

$$\dot{\gamma} = \sum_{2 \leq m+n \leq 2N+1} \Gamma_{m,n}(\lambda) y^m \bar{y}^n + l_\gamma^{(k)}(R_k) + \mathit{Remainder} \tag{60}$$

where $\Theta_{m,n}(\lambda) = 0$ for $m + n \leq N$ and $m \neq n + 1$, $l_\lambda^{(k)}$, $l_y^{(k)}$ and $l_\gamma^{(k)}$ are linear functionals of the first-order in y satisfying the estimates

$$|l_\lambda^{(k)}(g)|, \; |l_y^{(k)}(g)|, \; |l_\gamma^{(k)}(g)| \leq c|y| \|e^{-\epsilon_0|x|} g\|_2, \tag{61}$$

Remainder obeys the estimate in Equation (40). Moreover, if the functions $R_{m_1,n_1}(\lambda)$ are admissible for all pairs $(m_1, n_1) < (m, n)$ with $m, n \leq N$ and $m + n \leq k$, then $\Lambda_{m,n}(\lambda)$ and $\Theta_{m,n}(\lambda)$ are purely imaginary and $\Gamma_{m,n}(\lambda)$ are real.

We prove this lemma in Appendix A. Meantime we proceed with the proof of Theorem 7.2. We determine the coefficients $R_{m,n}(\lambda)$, $m+n \leq k$, by solving the equations $F_{mn}(\lambda) = 0$, i.e.

$$[L(\lambda) - i\epsilon(\lambda)(m-n)]R_{mn}(\lambda) = -P_c f_{mn}(\lambda). \tag{62}$$

By Lemma 7.4 we have that $R_{m,n}(\lambda)$ solving Equation (62) is admissible for $m, n \leq N$, (and hence $|m-n|\epsilon(\lambda) < \lambda$), if so is $if_{m,n}(\lambda)$. By Property (A) in Lemma 7.9, $if_{m,n}(\lambda)$ is admissible if so are $R_{m',n'}(\lambda)$ with $(m', n') < (m, n)$. Thus if $R_{m',n'}(\lambda)$ is admissible for every $(m', n') < (m, n)$, then so is $R_{m,n}(\lambda)$. Since $R_{m,n}(\lambda),\ m+n \leq N$ are admissible, we have by induction in (m', n') that $R_{m,n}(\lambda),\ m, n \leq N$, are admissible. This proves (34) with (RA) and (RB). Property (RC) follows from Lemma 7.9 and the equations $F_{m,n}(\lambda) = 0,\ 2 \leq m+n \leq k$.

Furthermore, when $k = 2N$, we have by (61) above that

$$|l_{\lambda}^{(2N)}(R_{2N})|,\ |l_{y}^{(2N)}(R_{2N})|,\ |l_{\gamma}^{(2N)}(R_{2N})| \leq c|y|\|e^{-\epsilon_0|x|}R_{2N}\|_2.$$

Moreover, since

$$|y(t)|\|e^{-\epsilon_0|x|}R_{2N}(t)\|_2 \leq c(1+t)^{-\frac{2N+2}{2N}}Y(t)\mathcal{R}_1(t),$$

where the estimating functions Y and $\mathcal{R}_1$ are defined in (41), the terms $l_{\lambda}^{(2N)}(R_{2N}),\ l_{y}^{(2N)}(R_{2N})$ and $l_{\gamma}^{(2N)}(R_{2N})$ obey the estimates in (61) and therefore can be placed into *Remainder*. Hence the equations for $\dot{y}$, $\dot{\lambda}$ and $\dot{\gamma}$ in Lemma 7.9 imply the corresponding equations given in (39) and Theorem 7.3.

□

8 Estimates on λ

In this section we obtain an estimate which, together with estimates on $Y(T)$ and $R_j(T),\ j = 1, 2, 3$, obtained in Section 10, will imply the convergence of the parameter $\lambda(t)$ as $t \to \infty$.

Proposition 8.1. *There exists a constant c such that for any t and T such that* $t \leq T$

$$|\lambda(t) - \lambda(T)| \leq c(T_0 + t)^{-\frac{1}{2N}}(Y(T) + \mathcal{R}_1(T))^2. \tag{63}$$

Proof. First we note that Equation (42) does not imply directly Estimate (63). To obtain (63) we transform y as

Proposition 8.2. *There exists a transformation y to β s.t.* $\beta = y + O(|y|^2)$ *and*

$$\frac{d}{dt}[\lambda - \sum_{2\leq m+n\leq 2N+1} a_{m,n}(\lambda)\beta^m\bar{\beta}^n] = Remainder, \tag{64}$$

where, $a_{m,n}(\lambda): \mathbb{R}^+ \to \mathbb{C}$ *and the Remainder satisfies Estimate (40).*

This proposition will be proved in Subsection 8.1. By Proposition 8.2 we have

$$\begin{array}{rl} & |\lambda(t) - \sum_{2\leq m+n\leq 2N+1} a_{m,n}(\lambda(t))\beta^m\bar{\beta}^n(t) \\ & -\lambda(T) + \sum_{2\leq m+n\leq 2N+1} a_{m,n}(\lambda(T))\beta^m\bar{\beta}^n(T)| \\ = & |\int_t^T Remainder(s)ds|. \end{array}$$

By the estimate of *Remainder* in (40) we have that for any $t \leq T$

$$|\int_t^T Remainder(s)ds| \leq c(T_0+t)^{-\frac{1}{N}}(Y(T)+\mathcal{R}_1(T))^2.$$

By the definition of Y in Equation (41) and the fact that $\beta = y + O(|y^2|)$ we have $|\beta(t)| \leq c(1+t)^{-\frac{1}{2N}}Y(t)$ for some constant c. Therefore (63) follows. □

8.1 Proof of Proposition 8.2

Below *Remainder* **signifies a function satisfying (40).** We begin with

Lemma 8.3. *There exists a polynomial*

$$P_1(y,\bar{y}) = \sum_{N+1\leq m+n\leq 2N+1} u_{mn}(\lambda)y^m\bar{y}^n$$

where the coefficients $u_{m,n}$ are real for $m,n \leq N$, such that if we let

$$\beta := y + P_1(y,\bar{y})$$

then

(A) $\dot{\beta} = i\epsilon(\lambda)\beta + \sum_{1\leq n\leq N} Y_n(\lambda)\beta^{n+1}\bar{\beta}^n + Remainder$*, with the coefficients* $Y_n(\lambda)$ *purely imaginary for $n < N$;*

(B) $\dot{\lambda} = \sum_{2\leq m+n\leq 2N+1} \lambda_{m,n}(\lambda)\beta^m\bar{\beta}^n + Remainder$*, with $\lambda_{m,n}(\lambda)$ purely imaginary for any $m,n \leq N$.*

The proof of this lemma is given in Subsection 8.2. Note that Statement (A) is the same as Statement (B) of Main Theorem 5.2. We prove Proposition 8.2 by using inductions on the number $k = m+n$. Suppose that for $1 \leq k < 2N+1$

$$\frac{d}{dt}[\lambda - \sum_{2\leq m+n\leq k} a_{mn}(\lambda)\beta^m\bar{\beta}^n] = \sum_{k+1\leq m+n\leq 2N+1} b_{m,n}(\lambda)\beta^m\bar{\beta}^n + Remainder,$$

where $b_{m,n}(\lambda)$ are purely imaginary for $m,n \leq N$. This latter properties together with the fact that λ is real imply that $b_{nn}(\lambda) = 0$. Since $a_{m,n}(\lambda) = 0$ for $m+n = 1$, the first step of induction, $k = 1$, is automatically true.

To remove the leading order from the right hand side of the last equation we rewrite it as

$$\begin{aligned} & \tfrac{d}{dt}[\lambda - \sum_{m+n\leq k} a_{mn}(\lambda)\beta^m\bar{\beta}^n - \sum_{m+n=k+1} \frac{b_{m,n}(\lambda)}{i(m-n)\epsilon(\lambda)}\beta^m\bar{\beta}^n] \\ = \quad & B_{k+1} + \sum_{k+2\leq m+n\leq 2N+1} b_{m,n}(\lambda)\beta^m\bar{\beta}^n + Remainder. \end{aligned} \tag{65}$$

where

$$B_{k+1} := \sum_{m+n=k+1} b_{m,n}(\lambda)\beta^m\bar{\beta}^n - \frac{d}{dt}\sum_{m+n=k+1}\frac{b_{m,n}(\lambda)}{i(m-n)\epsilon(\lambda)}\beta^m\bar{\beta}^n. \tag{66}$$

Then the right hand side of Equation (65) is of order $|\beta|^{k+2}$.

By the $k-$ step assumption the second term on the right hand side of (65) is of the form required by the $(k+1)-$step of the induction. Now we show that the first term on the right hand side, B_{k+1}, is also of the right form, i.e.

$$B_{k+1} = \sum_{k+1\leq m+n\leq 2N+1} c_{m,n}(\lambda)\beta^m\bar{\beta}^n + Remainder$$

where the coefficients $c_{m,n}(\lambda)$ are purely imaginary for $m, n \leq N$. Indeed, we expand the term (66) as

$$\begin{aligned} B_{k+1} \quad = \quad & -\sum_{m+n=k+1}\dot{\lambda}\frac{d}{d\lambda}(\frac{b_{m,n}(\lambda)}{i(m-n)\epsilon(\lambda)})\beta^m\bar{\beta}^n \\ & -\sum_{m+n=k+1}\frac{b_{m,n}(\lambda)}{i(m-n)\epsilon(\lambda)}(\frac{d}{dt}\beta^m\bar{\beta}^n - i(m-n)\epsilon(\lambda)\beta^m\bar{\beta}^n) + Remainder \\ = \quad & -\sum_{2\leq m'+n'\leq 2N+1}\sum_{m+n=k+1}\lambda_{m'n'}(\lambda)\frac{d}{d\lambda}(\frac{b_{m,n}(\lambda)}{i(m-n)\epsilon(\lambda)})\beta^{m+m'}\bar{\beta}^{n+n'} \\ & -\sum_{1\leq n'\leq N}\sum_{m+n=k+1}[m\frac{b_{m,n}Y_{n'}(\lambda)}{i(m-n)\epsilon(\lambda)} + n\frac{b_{m,n}\bar{Y}_{n'}(\lambda)}{i(m-n)\epsilon(\lambda)}]\beta^{m+n'}\bar{\beta}^{n+n'} \\ & +Remainder. \end{aligned} \tag{67}$$

By the properties of $b_{m,n}, \lambda_{m,n}$ and Y_n, we have that if $m+m',\ n+n' \leq N$ then $\lambda_{m'n'}(\lambda)\frac{d}{d\lambda}(\frac{b_{m,n}(\lambda)}{i(m-n)\epsilon(\lambda)})$ is purely imaginary; if $m+n', n+n' \leq N$ then $m\frac{b_{m,n}Y_{n'}(\lambda)}{i(m-n)\epsilon(\lambda)}$ and $n\frac{b_{m,n}\bar{Y}_{n'}(\lambda)}{i(m-n)\epsilon(\lambda)}$ are purely imaginary.

Thus we proved that

$$\begin{aligned} & \tfrac{d}{dt}[\lambda - \sum_{2\leq m+n\leq k} a_{mn}(\lambda)\beta^m\bar{\beta}^n - \sum_{m+n=k+1}\frac{b_{m,n}(\lambda)}{i(m-n)\epsilon(\lambda)}\beta^m\bar{\beta}^n] \\ = \quad & \sum_{k+2\leq m+n\leq 2N+1} b^{(1)}_{m,n}(\lambda)\beta^m\bar{\beta}^n + Remainder \end{aligned}$$

where the coefficients $b^{(1)}_{m,n}(\lambda)$ are purely imaginary for $m, n \leq N$. Thus the induction is complete. Taking $k = 2N+1$ yields Equation (64).

□

8.2 Proof of Lemma 8.3

Below *Remainder* **signifies a term satisfying (40).** We prove Statement (A) by induction. We define a set

$$\mathcal{A}_k := \{(m,n) | m,n \in \mathbb{Z}^+,\ m+n=k,\ m \neq n+1\}. \tag{68}$$

Suppose that for $N < k \leq 2N+1$ we found a transformation $\beta_k = y + P_1^{(k)}(y,\bar{y})$ such that β_k satisfies the equation

$$\begin{aligned} \dot{\beta}_k &= i\epsilon(\lambda)\beta_k + \sum_{n=1}^{N} \Theta_n(\lambda)\beta_k^{n+1}\bar{\beta}_k^n + \sum_{k\leq l\leq 2N+1}\sum_{(m,n)\in\mathcal{A}_l} \Theta_{m,n}(\lambda)\beta_k^m\bar{\beta}_k^n \\ &\quad + Remainder \end{aligned}$$

where $\Theta_n(\lambda) \equiv \Theta_{n,n}(\lambda)$ are purely imaginary if $n < N$ and $\Theta_{m,n}(\lambda)$ are purely imaginary for $m,n \leq N$. Note that by (39) when $k = N+1$ the equation above holds for $\beta_k = y$. Thus we have the first step of the induction.

We have that

$$\frac{d}{dt}\beta_{k+1} = i\epsilon(\lambda)\beta_{k+1} + \sum_{n=1}^{N}\Theta_n(\lambda)\beta_{k+1}^{n+1}\bar{\beta}_{k+1}^n + D_1 + D_2 + D_3 + Remainder, \tag{69}$$

where the new function β_{k+1} is defined as

$$\beta_{k+1} := \beta_k - \sum_{(m,n)\in\mathcal{A}_k} \frac{\Theta_{m,n}(\lambda)}{i(m-n-1)\epsilon(\lambda)}\beta_k^m\bar{\beta}_k^n, \tag{70}$$

and we observe that $\frac{\Theta_{m,n}(\lambda)}{i(m-n-1)\epsilon(\lambda)}$ are real for $m,n \leq N$; the terms D_n, $n = 1,2,3$, are given by

$$D_1 := \sum_{k+1\leq l\leq 2N+1}\sum_{(m,n)\in\mathcal{A}_l}\Theta_{m,n}(\lambda)\beta_k^m\bar{\beta}_k^n,$$

$$D_2 := \sum_{n=1}^{N}\Theta_n(\lambda)\beta_k^{n+1}\bar{\beta}_k^n - \sum_{n=1}^{N}\Theta_n(\lambda)\beta_{k+1}^{n+1}\bar{\beta}_{k+1}^n,$$

$$D_3 := -\frac{d}{dt}\sum_{(m,n)\in\mathcal{A}_k}\frac{\Theta_{m,n}(\lambda)}{i(m-n-1)\epsilon(\lambda)}\beta_k^m\bar{\beta}_k^n + \sum_{(m,n)\in\mathcal{A}_k}\frac{(m-n)\Theta_{m,n}(\lambda)}{m-n-1}\beta_k^m\bar{\beta}_k^n.$$

By Proposition B.1 in Appendix B

$$\begin{aligned} \dot{\beta}_{k+1} &= i\epsilon(\lambda)\beta_{k+1} + \sum_{n=1}^{N}\Theta_n^{(1)}(\lambda)\beta_{k+1}^{n+1}\bar{\beta}_{k+1}^n + \sum_{2\leq l\leq 2N+1}\sum_{(m,n)\in\mathcal{A}_l}\Theta_{m,n}^{(1)}(\lambda)\beta_{k+1}^m\bar{\beta}_{k+1}^n \\ &\quad + Remainder \end{aligned} \tag{71}$$

with $\Theta^{(1)}_{m,n}(\lambda)$ being purely imaginary if $m,n\leq N$ and $\Theta^{(1)}_{n}(\lambda)$ are purely imaginary if $n<N$. We claim that $\Theta^{(1)}_{m,n}(\lambda)=0$ for $(m,n)\in\cup_{l\leq k}\mathcal{A}_l$. This is due to the fact that the terms $D_n,\ n=1,2,3$, in Equation (69) are of the order $O(|\beta|^{k+1})$. This relations together with Equation (71) imply

$$\begin{aligned}\dot{\beta}_{k+1} &= i\epsilon(\lambda)\beta_{k+1}+\sum_{n=1}^{N}\Theta^{(1)}_{n}(\lambda)\beta^{n+1}_{k+1}\bar{\beta}^{n}_{k+1}+\sum_{k+1\leq l\leq 2N+1}\sum_{(m,n)\in\mathcal{A}_l}\Theta^{(1)}_{m,n}(\lambda)\beta^{m}_{k+1}\bar{\beta}^{n}_{k+1}\\ &\quad +Remainder\end{aligned}$$

where $\Theta^{(1)}_{m,n}(\lambda)$ are purely imaginary for $m,n\leq N$ and $\Theta^{(1)}_{n}(\lambda)$ are purely imaginary for $n<N$. Thus we complete the induction steps. Taking $\beta=\beta_{2N+1}$ we see that β satisfies the statement (A) of Lemma 8.3.

Now we prove Statement (B). By Statement (A)

$$\beta=y+\sum_{N+1\leq m+n\leq 2N+1}u_{mn}(\lambda)y^m\bar{y}^n \tag{72}$$

with $u_{mn}(\lambda)$ being real for $m,n\leq N$. We invert this function to get the relation

$$y=\beta-\sum_{N+1\leq m+n\leq 2N+1}u_{mn}(\lambda)\beta^m\bar{\beta}^n+Remainder$$

where $u_{m,n}(\lambda)$ are the same as in (72). We substitute the expression for y in Equation (42) to obtain

$$\dot{\lambda}=\sum_{2\leq m+n\leq 2N+1}\lambda_{mn}(\lambda)\beta^m\bar{\beta}^n+Remainder$$

with

$$\lambda_{m,n}(\lambda):=\Lambda_{m,n}(\lambda)-\sum_{\substack{m'+l_1=m+1\\ n'+l_2=n}}l_1u_{m',n'}\Lambda_{l_1,l_2}(\lambda)-\sum_{\substack{m'+l_2=m\\ n'+l_1=n+1}}l_2u_{m',n'}\bar{\Lambda}_{l_1,l_2}(\lambda).$$

If $m,n\leq N,\ l_1\neq 0,\ m'+l_1=m+1,\ n'+l_2=n$ and $m'+n'\geq N+1$ then $l_1,l_2,m',n'\leq N$. Thus $l_1u_{m',n'}\Lambda_{l_1,l_2}(\lambda)$ in the equation above are purely imaginary, where, recall the property of $\Lambda_{m,n}(\lambda)$ from (42) if $m'+l_1-1=m,\ n'+l_2=n\leq N$. Similarly $l_2u_{m',n'}\bar{\Lambda}_{l_1,l_2}(\lambda)$ is purely imaginary if $m'+l_2=m,\ n'+l_1-1=n\leq N$. Therefore $\lambda_{m,n}(\lambda)$ is purely imaginary for $m,n\leq N$.

□

9 The Decay of y

Let the parameter β be the same as in Lemma 8.3. Recall that $ReY_N(\lambda)<0$ by Condition (FGR) in Theorem 5.1. We have

Lemma 9.1. *for any $t \leq T$ we have*

$$|y|, |\beta| \leq c(T_0+t)^{-\frac{1}{2N}}[1+T_0^{-\frac{1}{2N}}(Y(T)+\mathcal{R}_1(T))^2] \tag{73}$$

for some constant c.

Proof. For any $t \geq 0$, define

$$X(t) := \sup_{s\leq t}(T_0+s)^{\frac{1}{2N}}|\beta(s)|.$$

By the relationship between β and y we have that if X is uniformly bounded in t, then

$$cY \leq X \leq \frac{1}{c}Y \tag{74}$$

for some constant c, where, recall the functions $Y = Y(t), \mathcal{R}_n = \mathcal{R}_n(t),\ n = 1,2,3$, defined in Equation (87). We claim that

$$X \leq cX(0)[1+(Y+\mathcal{R}_1)^2]. \tag{75}$$

Indeed, by the equation in Statement (A) of Lemma 8.3 we have that

$$\frac{1}{2}\frac{d}{dt}|\beta|^2 = ReY_N(\lambda)|\beta|^{2N+2} + Re(\bar{\beta}\text{Remainder}) \tag{76}$$

which can be transformed into a Riccati equation

$$\begin{array}{rcl} \frac{1}{2N}\frac{d}{dt}|\beta|^{2N} & = & ReY_N(\lambda)|\beta|^{4N} + |\beta|^{2N-2}Re(\bar{\beta}\text{Remainder}) \\ & \leq & ReY_N(\lambda)|\beta|^{4N} + |\beta|^{2N-1}|Remainder|. \end{array} \tag{77}$$

By the estimate of *Remainder* in Equation (40), the property $ReY_N(\lambda) < 0$ (see Condition (FGR)) and Equations (74) and (77) we have Equation (75). This together with Equation (74) implies Lemma 9.1. □

10 Proof of the Main Theorems 5.1 and 5.2 for $d \geq 3$

In order not to complicate notations we construct the proof of the main Theorems 5.1 and 5.2 for $d = 3$ rather than $d \geq 3$. This proof can be easily modified to obtain the general $d \geq 3$ cases (the only difference is that one has to deal with $[\frac{d}{2}]+3$ derivatives, see Subsection 10.1). We begin with some preliminary results. The following lemma will be used repeatedly.

Lemma 10.1. *There is a constant $\epsilon > 0$ such that if $|\lambda-\lambda_1| \leq \epsilon$ then there is a constant $c > 0$ such that*

$$\begin{array}{c} \|\rho_\nu(-\Delta+1)^2R_N\|_2 \leq c\|\rho_\nu(-\Delta+1)^2P_c^{\lambda_1}R_N\|_2, \\ \|R_N\|_\infty \leq c\|P_c^{\lambda_1}R_N\|_\infty, \\ \|R_N\|_{\mathcal{H}^2} \leq c\|P_c^{\lambda_1}R_N\|_{\mathcal{H}^2}, \end{array} \tag{78}$$

$$\|\rho_\nu R_{2N}\|_2 \leq c\|\rho_\nu P_c^{\lambda_1}R_{2N}\|_2. \tag{79}$$

Proof. We only prove the first three estimates, the proof of (79) is similar. First, since the vectors

$$\xi_1 := \begin{pmatrix} 0 \\ \phi^{\lambda_1} \end{pmatrix}, \ \xi_2 := \begin{pmatrix} \frac{d}{d\lambda}\phi^{\lambda_1} \\ 0 \end{pmatrix}, \ \xi_3 := \begin{pmatrix} \xi^{\lambda_1} \\ 0 \end{pmatrix}, \ \xi_4 := \begin{pmatrix} 0 \\ \eta^{\lambda_1} \end{pmatrix}$$

span the space $Range\{1 - P_c^{\lambda_1}\}$ there exists a vector $\vec{a} = (a_1, \cdots, a_4)$ such that

$$R_N = P_c^{\lambda_1} R_N + \sum_{n=1}^{4} a_n \xi_n. \tag{80}$$

From the equation $(1 - P_c^{\lambda})R_N = 0$ we derive the equation $A\vec{a} = -\vec{b}$ where $\vec{b}$ is a 4×1 vector with the components $b_j := \langle P_c^{\lambda_1} R_N, \xi_j \rangle$, and A is the 4×4 matrix

$$A := \begin{pmatrix} 0 & \langle \phi^{\lambda}, \frac{d}{d\lambda}\phi^{\lambda_1}\rangle & \langle \phi^{\lambda}, \xi^{\lambda_1}\rangle & 0 \\ \langle \phi^{\lambda_1}, \frac{d}{d\lambda}\phi^{\lambda}\rangle & 0 & 0 & \langle \eta^{\lambda_1}, \frac{d}{d\lambda}\phi^{\lambda_1}\rangle \\ 0 & \langle \frac{d}{d\lambda}\phi^{\lambda_1}, \eta^{\lambda}\rangle & \langle \xi^{\lambda_1}, \eta^{\lambda}\rangle & 0 \\ \langle \phi^{\lambda_1}, \xi^{\lambda}\rangle & 0 & 0 & \langle \eta^{\lambda_1}, \xi^{\lambda}\rangle \end{pmatrix}.$$

By the fact that $\lambda \in \mathcal{I}$ (the interval $\mathcal{I}$ is defined in Equation (5)) and Equation (15) we have

$$\langle \phi^{\lambda}, \frac{d}{d\lambda}\phi^{\lambda}\rangle, \ \langle \xi^{\lambda}, \eta^{\lambda}\rangle \geq c > 0$$

for some constant c and by (14)

$$\langle \phi^{\lambda}, \xi^{\lambda}\rangle = \langle \frac{d}{d\lambda}\phi^{\lambda}, \eta^{\lambda}\rangle = 0.$$

Thus if $|\lambda - \lambda_1|$ is small, then the matrix A is invertible and $\|A^{-1}\| \leq C$ for some constant C. Thus $\vec{a} = -A^{-1}\vec{b}$ and therefore $|\vec{a} \leq c|\vec{b}|$. By Equation (80) and the definition of $\vec{b}$ we have

$$\|(1 - P_c^{\lambda_1})R_N\| \leq c\|P_c^{\lambda_1} R_N\|$$

in the spaces $\rho_{-\nu}\mathcal{H}^4$, $\mathcal{H}^2$ and $\mathcal{L}_\infty$. Thus

$$\|R_N\| \leq \|P_c^{\lambda_1} R_N\| + \|(1 - P_c^{\lambda_1})R_N\| \leq (c+1)\|P_c^{\lambda_1} R_N\|$$

which is Equation (78). $\square$

10.1 Estimates on the Propagator

We will need the following estimates of the evolution operator $U(t) := e^{tL(\lambda_1)}$ where $\lambda_1 := \lambda(T)$ for some fixed $T \geq 0$, which we formulate in the general case $d \geq 3$ though we consider presently only the case $d = 3$:

$$\|\rho_\nu(-\Delta + 1)^k U(t) P_c^{\lambda_1} h\|_2 \leq c(1+t)^{-\frac{d}{2}} \|\rho_{-\nu}(-\Delta + 1)^k h\|_2; \tag{81}$$

$$
\begin{array}{rl}
& \|\rho_\nu(-\Delta+1)^k \prod(L(\lambda)-ik\epsilon(\lambda)+i0)^{-n_k}U(t)P_c^{\lambda_1}h\|_2 \\
\leq & c(1+t)^{-d/2}\|e^{\epsilon|x|}(-\Delta+1)^k h\|_2
\end{array} \tag{82}
$$

with $\sum n_k \leq 2N$;

$$
\|U(t)P_c^{\lambda_1}h\|_{\mathcal{L}^\infty} \leq ct^{-d/2}\|h\|_1; \tag{83}
$$

$$
\|U(t)P_c^{\lambda_1}h\|_\infty \leq c(1+t)^{-d/2}(\|h\|_{\mathcal{H}^k}+\|h\|_1); \tag{84}
$$

$$
\|\rho_\nu(-\Delta+1)^k U(t)P_c^{\lambda_1}h\|_2 \leq c(1+t)^{-d/2}(\|(-\Delta+1)^k h\|_1+\|(-\Delta+1)^k h\|_2) \tag{85}
$$

where ϵ is any positive constant, $k := [\frac{d}{2}]+1$ and ν is a large positive constant depending on N. Estimate (81) comes from the estimate

$$
\|\rho_\nu U(t)P_c^{\lambda_1}h\|_2 \leq c(1+t)^{-\frac{d}{2}}\|\rho_{-\nu}h\|_2
$$

proved in [RSS], and the observation that

$$
\begin{array}{rcl}
\|\rho_\nu(-\Delta+1)^k U(t)P_c^{\lambda_1}h\|_2 & \leq & c\|\rho_\nu(L(\lambda_1)+k_1)^k U(t)P_c^{\lambda_1}h\|_2 \\
& = & c\|\rho_\nu U(t)P_c^{\lambda_1}(L(\lambda_1)+k_1)^k h\|_2
\end{array} \tag{86}
$$

for some constant k_1, and $k := [\frac{d}{2}]+1$. Estimate (83) can be proved by the same technique as in [GoSc] where a version of this estimate for the case of self-adjoint operators is proved. Estimates (84) and (85) follow from Estimate (83) (the long time part) and the estimate

$$
\|U(t)P_c^{\lambda_1}h\|_\infty \leq c\|U(t)P_c^{\lambda_1}h\|_{\mathcal{H}_k} \leq c\|h\|_{\mathcal{H}^k}
$$

and

$$
\|\rho_\nu(-\Delta+1)^k U(t)P_c^{\lambda_1}h\|_2 \leq \|(-\Delta+1)^k U(t)P_c^{\lambda_1}h\|_\infty
$$

(the short time part). Estimate (82) comes from Estimate (81) and the technique of deformation of contour of integration from [BuSu, Rauch, RSS].

In the next subsections we begin estimating the majorants $\mathcal{R}_n,\ n=1,2,$ and Y defined in Equation (41). We write

$$
\mathcal{R}_1 = \mathcal{R}_a+\mathcal{R}_b+\mathcal{R}_c
$$

where

$$
\begin{array}{rcl}
\mathcal{R}_a(T) & := & \max_{t\leq T}(T_0+t)^{\frac{N+1}{2N}}\|\rho_\mu R_N\|_{\mathcal{H}^l}, \\
\mathcal{R}_b(T) & := & \max_{t\leq T}(T_0+t)^{\frac{N+1}{2N}}\|\rho_\mu R_N\|_\infty, \\
\mathcal{R}_c(T) & := & \max_{t\leq T}(T_0+t)^{\frac{2N+1}{2N}}\|\rho_\mu R_{2N}\|_2,
\end{array} \tag{87}
$$

and recall the definitions of the constants l and T_0 after (41), and estimate the estimating functions $\mathcal{R}_a,\ \mathcal{R}_b,\ \mathcal{R}_c$ separately.

10.2 Estimate for $\mathcal{R}_a$

The following proposition is the main result of this subsection.

Proposition 10.2.

$$\mathcal{R}_a \le cT_0^{\frac{N+1}{2N}}\|\rho_{-2}R_N(0)\|_2+c(T_0^{-\frac{1}{2N}}Y\mathcal{R}_a+Y^{N+1}+Y^2\mathcal{R}_a^2\mathcal{R}_2^2+Y^2\mathcal{R}_a\mathcal{R}_b^2\mathcal{R}_2^3+\mathcal{R}_b^3\mathcal{R}_2^4].$$

Before proving the proposition we derive a new equation for R_N. If we write $L(\lambda(t)) = L(\lambda_1) + L(\lambda(t)) - L(\lambda_1)$, then Equation (36) for R_N takes the form

$$\frac{d}{dt}P_c^{\lambda_1}R_N = L(\lambda_1)P_c^{\lambda_1}R_N + (\lambda-\lambda_1+\dot{\gamma})P_c^{\lambda_1}\sigma_3 R_N + \cdots$$

where $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The propagator generated by the operator $L(\lambda_1) + (\lambda-\lambda_1+\dot{\gamma})P_c^{\lambda}\sigma_3$ is estimated using the following extension of a result from [BuSu] whose proof we omit. Denote by P_+ and P_- the projection operators onto the positive and negative branches of the essential spectrum of $L(\lambda_1)$, respectively. Then we have

Lemma 10.3. *For any function h we have*

$$\|\rho_{-\nu}(-\Delta+1)^2(P_c^{\lambda_1}\sigma_3 - iP_+ + iP_-)h\|_2 \le c\|\rho_\nu(-\Delta+1)^2 h\|_2$$

for any large $\nu > 0$.

Equation (36) can be rewritten as

$$\begin{aligned} \tfrac{d}{dt}P_c^{\lambda_1}R_N &= L(\lambda_1)P_c^{\lambda_1}R_N + [\dot{\gamma}+\lambda-\lambda_1]i(P_+-P_-)R_N \\ &\quad +P_c^{\lambda_1}O_1R_N + P_c^{\lambda_1}F_N(y,\bar{y}) + P_c^{\lambda_1}N_N(R_N,y,\bar{y}), \end{aligned} \tag{88}$$

where O_1 is the operator defined by

$$O_1 := P_N(y,\bar{y}) + \dot{\lambda}P_{c\lambda} + L(\lambda) - L(\lambda_1) + \dot{\gamma}P_c^{\lambda}\sigma_3 - [\dot{\gamma}+\lambda-\lambda_1]i(P_+-P_-) \tag{89}$$

and the definitions of and estimates on $P_N(y,\bar{y})$ and $N_N(R_N,y,\bar{y})$ are given in Theorem 7.2, Part (RC). Equations (41), (87), (63) and (73) imply that

$$\|\rho_{-\nu}(-\Delta+1)^2O_1R_N\|_2 \le c(T_0+t)^{-\frac{N+2}{2N}}Y\mathcal{R}_a, \tag{90}$$

$$\|\rho_{-\nu}(-\Delta+1)^2F_N(y,\bar{y})\|_2 \le c(T_0+t)^{-\frac{N+1}{2N}}Y^{N+1}, \tag{91}$$

$$\begin{aligned} &\|(-\Delta+1)^2N_N(R_N,y,\bar{y})\|_1 + \|(-\Delta+1)^2N_N(R_N,y,\bar{y})\|_2 \\ \le\ & c(T_0+t)^{-\frac{2N+3}{2N}}(Y^2\mathcal{R}_a^2\mathcal{R}_2^2 + Y^2\mathcal{R}_a\mathcal{R}_b^2\mathcal{R}_2^3 + \mathcal{R}_b^3\mathcal{R}_2^4). \end{aligned} \tag{92}$$

By Equation (88) and the observation that the operators P_+, P_- and $L(\lambda_1)$ commute with each other, we have

$$\begin{aligned} P_c^{\lambda_1}R_N &= e^{tL(\lambda_1)+a(t,0)(P_+-P_-)}P_c^{\lambda_1}R_N(0) \\ &\quad + \int_0^t e^{(t-s)L(\lambda_1)+a(t,s)(P_+-P_-)}P_c^{\lambda_1}[O_1R_N \\ &\quad +F_N(y,\bar{y}) + N_N(R_N,y,\bar{y})]ds, \end{aligned} \tag{93}$$

with $a(t,s) := \int_s^t i[\dot{\gamma}(k) + \lambda(k) - \lambda_1]dk$. We observe that $P_+P_- = P_-P_+ = 0$ and for any times $t_1 \le t_2$ the operator

$$e^{a(t_2,t_1)(P_+ - P_-)} = e^{a(t_2,t_1)}P_+ + e^{-a(t_2,t_1)}P_- : \mathcal{H}^4 \to \mathcal{H}^4$$

is uniformly bounded. Now we prove Proposition 10.2.
Proof of Proposition 10.2. By Equation (93), Estimates (81) and (85) for $d = 3$ we have

$$\begin{array}{rl} & \|\rho_\nu(-\Delta+1)^2 P_c^{\lambda_1} R_N(t)\|_2 \\ \le & \|\rho_\nu(-\Delta+1)^2 e^{tL(\lambda_1)} P_c^{\lambda_1} R_N(0)\|_2 \\ & +\| \int_0^t \rho_\nu(-\Delta+1)^2 e^{(t-s)L(\lambda_1)} P_c^{\lambda_1}[O_1(s)R_N + F_N(y,\bar{y}) + N_N(R_N,y,\bar{y})]ds\|_2 \\ \le & c(1+t)^{-3/2}\|\rho_{-\nu}(-\Delta+1)^2 R_N(0)\|_2 \\ & + \int_0^t (1+t-s)^{-3/2}\|\rho_{-\nu}(-\Delta+1)^2[O_1 R_N + F_N(y,\bar{y})]ds\|_2 \\ & + \int_0^t (1+t-s)^{-3/2}(\|(-\Delta+1)^2 N_N(R_N(s),y,\bar{y})\|_1 + \|(-\Delta+1)^2 N_N(R_N(s),y,\bar{y})\|_2)ds. \end{array} \tag{94}$$

Therefore by Lemma 10.1 and Estimates (90)-(92) we have

$$\begin{array}{rl} & \|\rho_\nu(-\Delta+1)^2 R_N\|_2 \\ \le & c_1\|\rho_\nu(-\Delta+1)^2 P_c^{\lambda_1} R_N\|_2 \\ \le & c_2[(1+t)^{-3/2}\|\rho_{-\nu}(-\Delta+1)^2 R_N(0)\|_2 + \int_0^t (1+t-s)^{-3/2}(T_0+s)^{-\frac{N+1}{2N}}ds \\ & \times(T_0^{-\frac{1}{2N}} Y\mathcal{R}_a + Y^{N+1} + Y^2\mathcal{R}_a^2\mathcal{R}_2^2 + Y^2\mathcal{R}_a\mathcal{R}_b^2\mathcal{R}_2^3 + \mathcal{R}_b^3\mathcal{R}_2^4]. \end{array}$$

Using the estimate

$$\int_0^t (1+t-s)^{-3/2}(T_0+s)^{-\frac{N+1}{2N}}ds \le c(T_0+t)^{-\frac{N+1}{2N}}$$

we obtain

$$\begin{array}{rcl} \|\rho_\nu(-\Delta+1)^2 R_N\|_2 & \le & c(T_0+t)^{-\frac{N+1}{2N}}[T_0^{\frac{N+1}{2N}}\|\rho_{-2}R_N(0)\|_2 + T_0^{-\frac{1}{2N}} Y\mathcal{R}_a + Y^{N+1} \\ & & +Y^2\mathcal{R}_a^2\mathcal{R}_2^2 + Y^2\mathcal{R}_a\mathcal{R}_b^2\mathcal{R}_2^3 + \mathcal{R}_b^3\mathcal{R}_2^4]. \end{array}$$

This and the definition of $\mathcal{R}_a$ (in Equation (87)) imply Proposition 10.2.

□

10.3 Estimate for $\mathcal{R}_b$

The following proposition is the main result of this subsection.

Proposition 10.4.

$$\mathcal{R}_b \le c[T_0^{\frac{N+1}{2N}}\|R_N(0)\|_1 + T_0^{\frac{N+1}{2N}}\|R_N(0)\|_{\mathcal{H}^2} + T_0^{-\frac{1}{2N}} Y\mathcal{R}_a + Y^2\mathcal{R}_a^2\mathcal{R}_b^4 + \mathcal{R}_b^5\mathcal{R}_2^2].$$

Proof. By Estimate (84) in $d = 3$, Lemma 10.1 and Equation (88) we have that

$$\begin{array}{rl} & \|R_N(t)\|_\infty \\ \leq & c\|P_c^{\lambda_1} R_N(t)\|_\infty \\ \leq & c\|e^{tL(\lambda_1)} P_c^{\lambda_1} R_N(0)\|_\infty + \int_0^t \|e^{(t-s)L(\lambda_1)} P_c^{\lambda_1}[O_1(s)R_N + F_N(y,\bar{y}) + N_N(R_N, y, \bar{y})]\|_\infty ds \\ \leq & c(1+t)^{-3/2}(\|R_N(0)\|_1 + \|R_N(0)\|_{\mathcal{H}^2}) \\ & +c\int_0^t (1+t-s)^{-3/2}[\|O_1(s)R_N + F_N(y,\bar{y})\|_1 + \|O_1(s)R_N + F_N(y,\bar{y})\|_{\mathcal{H}^2}]ds \\ & +c\int_0^t (1+t-s)^{-3/2}(\|N_N(R_N,y,\bar{y})\|_1 + \|N_N(R_N,y,\bar{y})\|_{\mathcal{H}^2})ds. \end{array} \tag{95}$$

By the properties of O_1 (Equation (89)) and F_N (Equation (36)) we have

$$\|O_1(s)R_N + F_N(y,\bar{y})\|_1 + \|O_1(s)R_N + F_N(y,\bar{y})\|_{\mathcal{H}^2} \leq c(T_0+t)^{-\frac{N+1}{2N}} Y(T)\mathcal{R}_a(T).$$

By Equation (38)

$$\|N_N(R_N,y,\bar{y})\|_1 + \|N_N(R_N,y,\bar{y})\|_{\mathcal{H}^2} \leq c(T_0+t)^{-\frac{2N+3}{2N}}[Y^2\mathcal{R}_a^2\mathcal{R}_b^4 + \mathcal{R}_b^5\mathcal{R}_2^2].$$

Hence

$$\begin{array}{rcl} \|R_N(t)\|_\infty & \leq & c(T_0+t)^{-\frac{N+1}{2N}}[T_0^{\frac{N+1}{2N}}\|R_N(0)\|_1 \\ & & +T_0^{\frac{N+1}{2N}}\|R_N(0)\|_{\mathcal{H}^2} + T_0^{-\frac{1}{2N}} Y(t)\mathcal{R}_a(t) + Y\mathcal{R}_a^2\mathcal{R}_b^4(t) + \mathcal{R}_b^5\mathcal{R}_2^2(t)]. \end{array}$$

This estimate and the definition of $\mathcal{R}_b$ yield the proposition. □

10.4 Estimate for $\mathcal{R}_c$

The following is the main result of this subsection

Proposition 10.5. *Let the constant ν the same as in (81)-(82) with $d = 3$. Then*

$$\mathcal{R}_c \leq c[T_0^{\frac{2N+1}{2N}}\|\rho_{-\nu}R_N(0)\|_2 + T_0^{\frac{2N+1}{2N}}|y|^{N+1}(0)] + c(T_0^{-\frac{1}{2N}}Y\mathcal{R}_c + Y^{2N+1} + Y^2\mathcal{R}_a^2\mathcal{R}_b^4 + \mathcal{R}_b^5\mathcal{R}_2^2). \tag{96}$$

Proof. By the same techniques as we used in deriving Equation (88) we have the following equation

$$\begin{array}{rcl} \frac{d}{dt}P_c^{\lambda_1}R_{2N} & = & L(\lambda_1)P_c^{\lambda_1}R_{2N} + (\dot{\gamma}+\lambda-\lambda_1)i(P_+ - P_-)R_{2N} \\ & & +P(y,\bar{y})R_{2N} + P_c^{\lambda_1}F_{2N}(y,\bar{y}) + P_c^{\lambda_1}N_N(R_N,y,\bar{y}), \end{array} \tag{97}$$

where the operator $P(y,\bar{y})$ is defined as

$$P(y,\bar{y}) := P_c^{\lambda_1}P_{2N}(y,\bar{y}) - (\dot{\gamma}+\lambda-\lambda_1)i(P_+ - P_-) + P_c^{\lambda_1}(L(\lambda) - L(\lambda_1)),$$

and the terms $F_{2N}(y,\bar{y})$, $P_{2N}(z,\bar{z})$ and $N_N(R_N,y,\bar{y})$ are defined in Theorem 7.2.

Rewriting Equation (97) in the integral form using the Duhamel principle and using Lemma 10.1 we obtain

$$\begin{array}{rcl} \|\rho_\nu R_{2N}(t)\|_2 & \leq & c\|\rho_\nu P_c^{\lambda_1} R_{2N}(t)\|_2 \\ & \leq & c\|\rho_\nu e^{tL(\lambda_1)} P_c^{\lambda_1} R_{2N}(0)\|_2 + c\int_0^t \|\rho_\nu e^{(t-s)L(\lambda_1)} \\ & & \times[P(y,\bar{y})R_{2N} + P_c^{\lambda_1} F_{2N}(y,\bar{y}) + N_N(R_N,y,\bar{y})]\|_2 ds. \end{array} \tag{98}$$

We claim that

$$\|\rho_\nu e^{tL(\lambda_1)} P_c^{\lambda_1} R_{2N}(0)\|_2 \leq c(1+t)^{-3/2}(\|\rho_{-\nu} R_N(0)\|_2 + |z(0)|^{N+1}). \tag{99}$$

Indeed, by Equations (34) we have that

$$R_{2N} = R_N + \sum_{N+1\leq m+n\leq 2N} R_{m,n}(\lambda) y^m \bar{y}^n.$$

Therefore, displaying the time-dependent of R_k, λ and y,

$$\begin{array}{rl} & \|\rho_\nu e^{tL(\lambda_1)} P_c^{\lambda_1} R_{2N}(0)\|_2 \\ \leq & \|\rho_\nu e^{tL(\lambda_1)} P_c^{\lambda_1} R_N(0)\|_2 + \displaystyle\sum_{N+1\leq m+n\leq 2N} |y(0)|^{m+n} \|\rho_\nu e^{tL(\lambda_1)} P_c^{\lambda_1} R_{m,n}\|_2 \end{array}$$

By Property (RA) of $R_{m,n}(\lambda)$ given in Theorem 7.2 and by Estimates (81) and (82) with $d=3$ we have that

$$\|\rho_\nu e^{tL(\lambda_1)} P_c^{\lambda_1} R_{m,n}\|_2 \leq c(1+t)^{-3/2}.$$

For the second term of the right hand side of Equation (98), we have

$$\begin{array}{rl} & \int_0^t \|\rho_\nu e^{(t-s)L(\lambda_1)}[P(y,\bar{y})R_{2N} + P_c^{\lambda_1} F_{2N}(y,\bar{y}) + N_N(R_N,y,\bar{y})]\|_2 ds \\ \leq & \int_0^t (1+t-s)^{-3/2}(\|\rho_{-\nu} P(y,\bar{y})R_{2N}\|_2 + \|N_N(R_N,y,\bar{y})\|_1 + \|N_N(R_N,y,\bar{y})\|_2) ds \\ & + \int_0^t \|\rho_\nu e^{(t-s)L(\lambda_1)} P_c^{\lambda_1} F_{2N}(y,\bar{y})\|_2 ds. \end{array} \tag{100}$$

For the terms on the right hand side of Equation (100) we have the following estimates:

(A) By the definition of $F_{2N}(y,\bar{y})$ in Equation (36) and Estimate (82) with $d=3$ we have that

$$\begin{array}{rl} & |\int_0^t \|\rho_\nu e^{(t-s)L(\lambda_1)} P_c^{\lambda_1} F_{2N}(y,\bar{y})\|_2 ds| \\ \leq & c_1 \int_0^t (1+t-s)^{-3/2}(T_0+s)^{-\frac{2N+1}{2N}} ds Y^{2N+1} \\ \leq & c_2(T_0+t)^{-\frac{2N+1}{2N}} Y^{2N+1}. \end{array}$$

(B) By Estimates (90)-(92) we have

$$\|N_N(R_N(s),y(s))\|_1 + \|N_N(R_N(s),y(s))\|_2 \leq c(T_0+s)^{-\frac{2N+1}{2N}}[Y^2\mathcal{R}_a^2\mathcal{R}_b^4 + \mathcal{R}_b^5\mathcal{R}_2^2].$$

(C) By the definition of $P(y,\bar{y})$ and the estimate of $P_{2N}(y,\bar{y})$ after Equation (36)

$$\begin{array}{rcl}\|\rho_{-\nu}P(y(s),\bar{y}(s))R_{2N}(s)\|_2 & \leq & c|y|\|\rho_\nu R_{2N}(s)\|_2 \\ & \leq & c(T_0+s)^{-\frac{2N+2}{2N}}Y\mathcal{R}_c.\end{array}$$

Collecting the estimates above we find

$$\begin{array}{rl} & \|\rho_\nu R_{2N}\|_2 \\ \leq & c(1+t)^{-3/2}[\|\rho_{-\nu}R_N(0)\|_2+|z|^{N+1}(0)]+c\int_0^t(1+t-s)^{-3/2}(T_0+s)^{-\frac{2N+1}{2N}}ds \\ & \times(T_0^{-\frac{1}{2N}}Y\mathcal{R}_c+Y^{2N+1}+Y^2\mathcal{R}_a^2\mathcal{R}_b^4+\mathcal{R}_b^5\mathcal{R}_2^2) \\ \leq & c(T_0+t)^{-\frac{2N+1}{2N}}[T_0^{\frac{2N+1}{2N}}\|\rho_{-\nu}R_N(0)\|_2+T_0^{\frac{2N+1}{2N}}|y|^{N+1}(0)+T_0^{-\frac{1}{2N}}Y\mathcal{R}_c+Y^{2N+1} \\ & +Y^2\mathcal{R}_a^2\mathcal{R}_b^4+\mathcal{R}_b^5\mathcal{R}_2^2].\end{array}$$

This and the definition of $\mathcal{R}_c$ yield (96). □

10.5 Estimate for $\mathcal{R}_2$

Proposition 10.6.

$$\mathcal{R}_2^2\leq\|R_N(0)\|_{\mathcal{H}^4}^2+c[\mathcal{R}_a^2+Y^2\mathcal{R}_a^2+Y^2\mathcal{R}_a^2\mathcal{R}_2+\mathcal{R}_2^5\mathcal{R}_b^3+Y^{N+1}\mathcal{R}_a]. \quad (101)$$

Proof. By Equation (31), we have

$$\begin{array}{rl} & \frac{d}{dt}\langle(-\Delta+1)^2R_N,(-\Delta+1)^2R_N\rangle \\ = & \langle(-\Delta+1)^2\frac{d}{dt}R_N,(-\Delta+1)^2R_N\rangle+\langle(-\Delta+1)^2R_N,(-\Delta+1)^2\frac{d}{dt}R_N\rangle \\ = & \sum_{n=1}^{4}K_n\end{array}$$

with

$K_1:=\langle(-\Delta+1)^2(L(\lambda)+\dot{\gamma}J)R_N,(-\Delta+1)^2R_N\rangle+\langle(-\Delta+1)^2R_N,(-\Delta+1)^2(L(\lambda)+\dot{\gamma}J)R_N\rangle;$

$K_2:=\dot{\lambda}\langle(-\Delta+1)^2P_{c\lambda}R_N,(-\Delta+1)^2R_N\rangle+\dot{\lambda}\langle(-\Delta+1)^2R_N,(-\Delta+1)^2P_{c\lambda}R_N\rangle;$

$K_3:=\langle(-\Delta+1)^2N_N(R_N,y,\bar{y}),(-\Delta+1)^2R_N\rangle+\langle(-\Delta+1)^2R_N,(-\Delta+1)^2N_N(R_N,y,\bar{y})\rangle;$

$K_4:=\langle(-\Delta+1)^2F_N(y,\bar{y}),(-\Delta+1)^2R_N\rangle+\langle(-\Delta+1)^2R_N,(-\Delta+1)^2F_N(y,\bar{y})\rangle.$

Recall the definition of $R_n,\ n=1,2,3$ and Y in (87).

Recall the definition of the operator $L(\lambda)$ in (10) and use the fact that $J^*=-J$ to obtain

$$|K_1|\leq c\|\rho_\nu R_N\|_{\mathcal{H}^4}^2\leq c(T_0+t)^{-\frac{2N+2}{2N}}\mathcal{R}_a^2.$$

By observing that $|\dot{\lambda}|=O(|y|^2)$ we have that

$$|K_2|\leq c|y|^2\|\rho_\nu R_N\|_{\mathcal{H}^4}^2\leq c(T_0+t)^{-\frac{2N+3}{2N}}Y^2(t)\mathcal{R}_a^2(t).$$

Moreover by the properties of $N_N(R_N, y, \bar{y})$ in (37) we have

$$|K_3| \leq c(T_0+t)^{-\frac{2N+2}{2N}}[Y^2\mathcal{R}_a^2\mathcal{R}_2(t)+\mathcal{R}_2^5\mathcal{R}_b^3(t)].$$

By the property of $F_N(y, \bar{y})$ in (36) we have

$$|K_4| \leq c|y|^{N+1}\|\rho_\nu R_N\|_{\mathcal{H}^4} \leq c(T_0+t)^{-\frac{2N+2}{2N}}Y^{N+1}\mathcal{R}_a.$$

Collecting all the estimates above we have

$$\begin{array}{rl} & |\frac{d}{dt}\langle(-\Delta+1)^2R_N, (-\Delta+1)^2R_N\rangle| \\ \leq & c(T_0+t)^{-\frac{2N+2}{2N}}[\mathcal{R}_a^2(t)+Y^2(t)\mathcal{R}_a^2(t)+Y^2\mathcal{R}_a^2\mathcal{R}_2(t)+\mathcal{R}_2^5\mathcal{R}_b^3(t)+Y^{N+1}\mathcal{R}_a] \end{array}$$

which implies that

$$\begin{array}{rcl} \|R_N(t)\|_{\mathcal{H}^4}^2 & \leq & \|R_N(0)\|_{\mathcal{H}^4}^2+c[\mathcal{R}_a^2(t) \\ & & +Y^2(t)\mathcal{R}_a^2(t)+Y^2\mathcal{R}_a^2\mathcal{R}_2(t)+\mathcal{R}_2^5\mathcal{R}_b^3(t)+Y^{N+1}\mathcal{R}_a(t)]. \end{array}$$

This and the definition of $\mathcal{R}_2$ implies (101). □

10.6 Proof of Main Theorems 5.1 and 5.2

Define $M(T) := \sum_{n=1}^{4}\mathcal{R}_n(T)$ and

$$S := T_0^{\frac{2N+1}{2N}}(\|R_N(0)\|_{\mathcal{H}^4}+\|\rho_{-\nu}R_N(0)\|_2+\|R_N(0)\|_1), \tag{102}$$

where, recall the definition of T_0 after (41). If $M(0)$ is sufficiently small and $Y(0)$ is bounded, then by Propositions 10.2, 10.4, 10.5, 10.6 and Equation (73) we obtain $M(T)+Y(T) \leq \mu(S)S$, where μ is a bounded function for S small. Thus we proved that if S and $M(0)$ are small, then

$$\|\rho_\nu R_N\|_2,\ \|R_N\|_\infty \leq c(T_0+t)^{-\frac{N+1}{2N}},\ |y(t)| \leq c(T_0+t)^{-\frac{1}{2N}} \tag{103}$$

for some constant c.

To complete the proof of Theorems 5.1 and 5.2 it suffices to show that

$$T_0^{\frac{2N+1}{2N}}(\|\vec{R}(0)\|_{\mathcal{H}^4}+\|\rho_{-\nu}\vec{R}(0)\|_2) \tag{104}$$

being small implies that S, defined in Equation (102), is small.

By Equation (34) we have

$$\begin{array}{rcl} S & \leq & cT_0^{\frac{2N+1}{2N}}[\|\vec{R}(0)\|_{\mathcal{H}^4}+\|\rho_{-\nu}\vec{R}(0)\|_2+\|\vec{R}(0)\|_1+|y^2(0)|] \\ & \leq & cT_0^{\frac{2N+1}{2N}}[\|\vec{R}(0)\|_{\mathcal{H}^4}+\|\rho_{-\nu}\vec{R}(0)\|_2+|y^2(0)|] \end{array} \tag{105}$$

for some constant $c>0$. Estimate (105) implies that if (104) is small, then S is small, and therefore Equation (103) holds. By Equations (34) and (103) we have

$$\|\rho_\nu\vec{R}(t)\| \leq c(T_0+t)^{-\frac{1}{N}} \text{ and } |y(t)| \leq c(T_0+t)^{-\frac{1}{2N}}.$$

Since (104) is small by the condition (19) on the datum, this together with the relationship $|z| = |y| + O(|y|^2)$, yields Statements (A) and (B) of Theorem 5.1.

Statement (A) of Theorem 5.2 follows from Propositions 8.1, 10.2, 10.4-10.6 by taking $T \geq t \to \infty$. Statement (B) is proved in Lemma 8.3.

□

11 Proof of Theorems 5.1 and 5.2 for $d = 1$

In this section we sketch the proof of Theorems 5.1 and 5.2 for the dimension 1. Most of the steps of the proof are almost the same to those of the case of $d \geq 3$, hence we concentrate on the parts which are different, namely, the rate of decay of $\|R_N\|_\infty$ where R_N is the remainder in the expansion of R in (34). Recall that $\rho_\nu := (1 + |x|)^{-\nu}$.

Theorem 11.1. *Let $d = 1$. Then the nonlinearity $N_N(R_N, y, \bar{y})$ in (36) satisfies, in addition, the estimate*

$$\begin{array}{ll} & \|\rho_{-2} N_N(R_N, y, \bar{y})\|_1 + \|N_N(R_N, y, \bar{y})\|_2 \\ \leq & c|y|^2 \|\rho_{\nu_1} R_N\|_2^2 \|R_N\|_\infty^{6N-1} + c\|(1 + |x|)R_N\|_2^2 \|R_N\|_\infty^{6N+1} \end{array} \tag{106}$$

for $\nu_1 > 7/2$ (see (108) below).

Now we prove the main theorems 5.1 and 5.2 for $d = 1$. We use Theorem 11.1 and Equation (98) to estimate R_N and R_{2N}. On the first sight we need estimates of the propagator generated by the time-dependent operator $L(\lambda(t))$. As in the $d \geq 3$ case, we use instead the estimates on the propagator $U(t) := e^{tL(\lambda_1)}$ where $\lambda_1 := \lambda(T)$ for some large fixed constant T. We have for $d = 1$

$$\|\rho_{\nu_1} U(t) P_c h\|_2 \leq c(1 + t)^{-\frac{3}{2}} \|\rho_{-2} h\|_2; \tag{107}$$

$$\|\rho_\nu \prod (L(\lambda) - ik_n \epsilon(\lambda) + i0)^{-n_{k_n}} U(t) P_c h\|_2 \leq c(1 + t)^{-3/2} \|e^{-\epsilon|x|} h\|_2 \tag{108}$$

with $\sum n_{k_n} \leq 2N$;

$$\|\rho_{\nu_1} U(t) P_c h\|_2 \leq c(1 + t)^{-3/2} (\|\rho_{-2} h\|_1 + \|h\|_2); \tag{109}$$

$$\|U(t) P_c h\|_{\mathcal{L}^\infty} \leq c t^{-1/2} \|\rho_{-2} h\|_2; \tag{110}$$

$$\|U(t) P_c h\|_{\mathcal{L}^\infty} \leq c t^{-1/2} (\|\rho_{-2} h\|_1 + \|h\|_2); \tag{111}$$

$$\|U(t) P_c h\|_{\mathcal{L}^\infty} \leq c(1 + t)^{-\frac{1}{2}} \|\rho_{-2} h\|_{\mathcal{H}^1}; \tag{112}$$

where $\epsilon > 0$, $\nu_1 > 7/2$, and ν is a large constant depending on N. Estimates (107) (109)-(112) were proved in [BP1, BuSu, GS1, Rauch]. To prove (108)

we use the technique of deformation of the contour as in the proof of (82) and [BuSu]. After fixing $L(\lambda(t))$ to be $L(\lambda_1)$ we have the equation

$$\frac{d}{dt}P_c^\lambda R_N = L(\lambda_1)P_c^\lambda R_N + (\lambda - \lambda_1 + \dot{\gamma})P_c^\lambda \sigma_3 R_N + \cdots.$$

To estimate the propagator $e^{t[L(\lambda_1)+(\lambda-\lambda_1+\dot{\gamma})P_c\sigma_3]}$ we use the following lemma similar to Lemma 10.3 (cf [BuSu]), whose proof we omit.

Lemma 11.2. *For any function h we have*

$$\|(1+x^2)(P_c^\lambda\sigma_3 - iP_+ + iP_-)h\|_2 \leq c\|\rho_\nu h\|_2$$

for any $\nu > 0$.

Equation (31) can be rewritten as

$$\begin{array}{rcl} \frac{d}{dt}P_c^\lambda R_N & = & L(\lambda_1)P_c^\lambda R_N + [\dot{\gamma}+\lambda-\lambda_1]i(P_+ - P_-)R_N \\ & & +P_c^\lambda O_1 R_N + P_c^\lambda F_N(y,\bar{y}) + P_c^\lambda N_N(R_N,y,\bar{y}), \end{array} \tag{113}$$

where, recall the definitions of and estimates on $F_N(y,\bar{y})$ and $N_N(R_N,y,\bar{y})$ given in Theorem 7.2 and Equation (106), and O_1 is the operator defined by

$$O_1 := A_2(z,\bar{z}) + \dot{\lambda}P_{c\lambda} + L(\lambda) - L(\lambda_1) + \dot{\gamma}P_c^\lambda\sigma_3 - [\dot{\gamma}+\lambda-\lambda_1]i(P_+ - P_-).$$

Note that for $d = 1$, $\|R_N\|_\infty$ has a slower decay rate. Hence we used different estimating functions than those used in Theorem 7.3. We replace the latter functions by the following estimating functions

$$\begin{array}{ll} \mathcal{R}_a(T) := \max\limits_{t\leq T}(T_0+t)^{\frac{N+1}{2N}}\|\rho_{\nu_1}R_N\|_2, & \mathcal{R}_b(T) := \max\limits_{t\leq T}(T_0+t)^{\frac{1}{2N}}\|R_N\|_\infty, \\ \mathcal{R}_c(T) := \max\limits_{t\leq T}(T_0+t)^{\frac{2N+1}{2N}}\|\rho_{\nu_2}R_{2N}(t)\|_2 & \end{array} \tag{114}$$

with $\nu_2 > 3.5$. The estimating function $Y(t)$ stays the same. (We use the same symbols since the estimating functions, $\mathcal{R}_n,\ n = 1,2,a,b,c$, defined in (41) and (87), are not used in this section.)

The next lemma is proved similarly to Equations (90), (91) in the $d \geq 3$ case.

Lemma 11.3.

$$\|\rho_{-2}O_1 R_N\|_2 \leq c(T_0+t)^{-\frac{N+2}{2N}}Y(T)\mathcal{R}_a(T),$$

$$\|\rho_{-2}F_2(y,\bar{y})\|_2 \leq c(T_0+t)^{-\frac{N+1}{2N}}Y^{N+1}.$$

11.1 Estimates of $\mathcal{R}_n,\ n = a,b,c,\ \lambda(t),\ y(t)$

In this subsection we will estimate the functions $\mathcal{R}_n$ which are defined in Equation (114), $\lambda(t)$ and $y(t)$.

Proposition 11.4. *Estimates (63) and (73) on $|\lambda(t)-\lambda(T)|$ and $|y(t)|$ hold in the case $d=1$ also. Moreover*

$$\begin{aligned}\mathcal{R}_a &\le cT_0^{\frac{N+1}{2N}}\|\rho_{-2}R_N(0)\|_2\\ &+c(T_0^{-\frac{1}{2N}}Y\mathcal{R}_a+Y^{N+1}+Y^2\mathcal{R}_a\mathcal{R}_b^{6N-1}+\mathcal{R}_b^{6N-1}+\mathcal{R}_b^{6N}),\end{aligned}$$

$$\begin{aligned}\mathcal{R}_c &\le cT_0^{\frac{2N+1}{2N}}[\|\rho_{-2}R_N(0)\|_2+|y|^{N+1}(0)]\\ &+c(T_0^{-\frac{1}{2N}}Y\mathcal{R}_c+Y^{2N+1}+Y^2\mathcal{R}_a\mathcal{R}_b^{6N-1}+\mathcal{R}_b^{6N-1}+\mathcal{R}_b^{6N}).\end{aligned}$$

$$\begin{aligned}\mathcal{R}_b &\le cT_0^{\frac{1}{2N}}\|\rho_{-2}R_N(0)\|_{\mathcal{H}^1}\\ &+c(T_0^{-\frac{1}{2N}}Y\mathcal{R}_a+Y^{N+1}+Y^2\mathcal{R}_a^2\mathcal{R}_b^{6N-3}+Y^2\mathcal{R}_a\mathcal{R}_b^{6N-2}+\mathcal{R}_b^{6N}+\mathcal{R}_b^{6N-1}).\end{aligned}$$

Proof. The estimates of $y(t)$, $|\lambda(t)-\lambda(T)|$, $\mathcal{R}_a$ and $\mathcal{R}_c$ are almost the same to the $d\ge 3$ case. Therefore we focus on the estimate of $\mathcal{R}_b$ which is different. (It is in this estimate where the condition (fC) for $d=1$ is used.)

By Lemma 10.1, an integral form of Equation (113) and Equations (110)-(112) we have that

$$\begin{aligned}\|R_N(t)\|_\infty &\le c\|P_c^\lambda R_N(t)\|_\infty\\ &\le c\|e^{tL(\lambda_1)}P_c^\lambda R_N(0)\|_\infty\\ &\quad+\int_0^t\|e^{(t-s)L(\lambda_1)}P_c^\lambda[O_1(s)R_N+F_2(y,\bar y)+N_N(R_N,y,\bar y)]\|ds\\ &\le c(1+t)^{-1/2}\|\rho_{-2}R_N(0)\|_{\mathcal{H}^1}\\ &\quad+\int_0^t(t-s)^{-1/2}\|\rho_{-2}[O_1(s)R_N+F_2(y,\bar y)]\|_2ds\\ &\quad+\int_0^t(t-s)^{-1/2}(\|\rho_{-2}N_N(R_N,y,\bar y)\|_1+\|N_N(R_N,y,\bar y)\|_2)ds.\end{aligned}\tag{115}$$

Recalling the estimates in Lemma 11.3, we obtain

$$\begin{aligned}\|R_N\|_\infty &\le c_1[(1+t)^{-1/2}\|\rho_{-2}R_N(0)\|_{\mathcal{H}^1}+\int_0^t(t-s)^{-1/2}(T_0+s)^{-\frac{N+1}{2N}}ds\\ &\times\ (T_0^{-\frac{1}{2N}}Y\mathcal{R}_a+Y^{N+1}+Y^2\mathcal{R}_a^2\mathcal{R}_b^{6N-3}+Y^2\mathcal{R}_a\mathcal{R}_b^{6N-2}+\mathcal{R}_b^{6N}+\mathcal{R}_b^{6N-1})].\end{aligned}$$

The proposition follows readily from this estimate, the easy inequality

$$\int_0^t(t-s)^{-1/2}(T_0+s)^{-1/2-\delta}ds\le c(T_0+t)^{-\delta}$$

valid for any $t\ge 0$ and $1/2>\delta>0$, and the definition of $\mathcal{R}_b$ in Equation (114). □

11.2 Proof of Main Theorems 5.1 and 5.2 for $d=1$

By Proposition 11.4 we have that if $T_0^{\frac{2N+1}{2N}}\|\rho_{-2}R(0)\|_{\mathcal{H}^1}$ is sufficiently small and $|Y(0)|$ is bounded, then $\mathcal{R}_n(T),\ n=a,b,c,\ Y(T)\le c$ for any time T. The rest of the proof of Theorems 5.1 and 5.2 is just repeat of the proof for the $d\ge 3$ case, given in Subsection 10.6.

□

A Proof of Lemma 7.9

Since the proof is long, we begin with Equations (58)-(60) first. Recall that $k > N$ in Lemma 7.9.

Lemma A.1. *(1) The linear functionals $l_\lambda^{(k)}$, $l_\gamma^{(k)}$ and $l_y^{(k)}$ satisfy the estimates (61).*

(2) If the functions $R_{m_1,n_1}(\lambda)$ are admissible for all pairs $(m_1, n_1) < (m, n)$ with $m, n \leq N$ and $m + n \leq k$, then $\Lambda_{m,n}(\lambda)$ and $\Theta_{m,n}(\lambda)$ in Equations (58) and (59) are purely imaginary, and $\Gamma_{m,n}(\lambda)$ in Equation (60) are real.

(3) $\Theta_{m,n}(\lambda) = Y_{m,n}(\lambda)$ if $m + n \leq N$, where, recall the definition of $Y_{m,n}(\lambda)$ and the property that $Y_{m,n}(\lambda) = 0$ if $m + n \leq N$ and $m \neq n + 1$ in (54).

Proof. (1) The estimates on $l_\lambda^{(k)}$, $l_\gamma^{(k)}$ and $l_y^{(k)}$ is easy to get by Equations (33) and (27) and the observations that the functions $\xi, \eta, \phi^\lambda, \phi_\lambda^\lambda$ decay exponentially fast.

(2) The proof of the properties of $\Lambda_{m,n}(\lambda)$ and $\Gamma_{m,n}(\lambda)$ are almost the same to those in the proof of Proposition 7.5, namely in all the computations only multiplications are involved, thus all pairs (m, n) depend only on the pairs $(m', n') < (m, n)$.

Now we turn to the proof for $\Theta_{m,n}(\lambda)$. Using Equation (32) we obtain

$$\dot{z} = i\epsilon(\lambda)z + \sum_{2\leq m+n\leq 2N+1} Y_{m,n}^{(2)}(\lambda)y^m\bar{y}^n + l_y^{(k)}(R_k) + Remainder \tag{116}$$

where $l_y^{(k)}$ satisfy the estimate (61), and *Remainder* satisfies Estimate (40). By the same arguments as was used for $\Lambda_{m,n}^{(2)}(\lambda)$ we obtain that $Y_{m,n}^{(2)}(\lambda)$, with $m, n \leq N$ and $m + n \leq k$, is purely imaginary if $R_{m',n'}(\lambda)$ are admissible for all the pairs $(m', n') < (m, n)$.

We invert the function $y = z + P(z, \bar{z})$ in Proposition 7.7 to get

$$z = y + \sum_{2\leq m+n\leq 2N+1} P_{m,n}^{(2)}(\lambda)y^m\bar{y}^n + Remainder \tag{117}$$

with $P_{m,n}^{(2)}(\lambda)$ being real. Plug this expression into Equation (116) to obtain

$$\dot{y} = i\epsilon(\lambda)y + D_1 + D_2 + l_y^{(k)}(R_k) + Remainder.$$

with

$$\begin{aligned} D_1 \quad := \quad & \sum_{2\leq m+n\leq 2N+1} Y_{m,n}^{(2)}(\lambda)y^m\bar{y}^n - \dot{\lambda} \sum_{2\leq m+n\leq 2N+1} \partial_\lambda P_{m,n}^{(2)}(\lambda)y^m\bar{y}^n \\ & -i\epsilon(\lambda)(m-n-1) \sum_{2\leq m+n\leq 2N+1} P_{m,n}^{(2)}(\lambda)y^m\bar{y}^n \end{aligned}$$

and

$$D_2 := -\sum_{2\le m+n\le 2N+1} P^{(2)}_{m,n}(\lambda)[\frac{d}{dt}y^m\bar{y}^n - i\epsilon(\lambda)(m-n)y^m\bar{y}^n].$$

Using (59), which is proved above, for the time derivatives in the expression for D_1 we obtain

$$D_1 = \sum_{2\le m+n\le 2N+1} D_{m,n}y^m\bar{y}^n + l_y^{(k)}(R_k) + Remainder,$$

where the functionals $l_y^{(k)}$ satisfy the estimate (61). We have that if the functions $R_{m_1,n_1}(\lambda)$ are admissible for all pairs $(m_1,n_1) < (m,n)$ with $m,n \le N$ and $m+n \le k$, then $D_{m,n}$ are purely imaginary. Indeed, this follows from the properties of the expansion for $\dot{\lambda}$ in (59), which is proved above, and by the properties of $P^{(2)}_{m,n}(\lambda)$ and $Y^{(2)}_{m,n}(\lambda)$ which we just mentioned (we omit the detail here).

Substitute in the expression for D_2 the equation (58) to get

$$\begin{aligned} D_2 \;=\; & \sum_{2\le m'+n'\le 2N+1,\ 2\le l_1+l_2\le 2N+1} \sum m' P^{(2)}_{m',n'}(\lambda)\Theta_{l_1,l_2}(\lambda) y^{m'+l_1-1}\bar{y}^{n'+l_2} \\ & + \sum_{2\le m'+n'\le 2N+1,\ 2\le l_1+l_2\le 2N+1} \sum n' P^{(2)}_{m',n'}\bar{\Theta}_{l_1,l_2}(\lambda) y^{m'+l_2}\bar{y}^{n'+l_1-1} \\ & + Remainder. \end{aligned} \tag{118}$$

We have that if $m'+l_1-1, n'+l_2 \le N$ then

$$\text{either } m'P^{(2)}_{m',n'}\Theta_{l_1,l_2}(\lambda) = 0 \text{ or } (m'+l_1-1, n'+l_2) > (l_1,l_2),$$

where, recall that $P^{(2)}_{m',n'}$ are real in (117). Thus if $m'+l_1-1,\ n'+l_2 \le N$ then $m'P^{(2)}_{m',n'}(\lambda)\Theta_{l_1,l_2}(\lambda)$ is purely imaginary if Θ_{l_1,l_2} is purely imaginary for all pairs $(l_1,l_2) < (m'+l_1-1, n'+l_2)$. The same results hold also for $n'P^{(2)}_{m',n'}\bar{\Theta}_{l_1,l_2}(\lambda)$. Thus if we expand

$$D_2 = \sum_{2\le m+n\le 2N+1} Y^{(4)}_{m,n}(\lambda)y^m\bar{y}^n + Remainder,$$

then $Y^{(4)}_{m,n}(\lambda)$, $m,n\le N$, are purely imaginary if $\Theta_{m',n'}(\lambda)$ are purely imaginary for all pairs $(m',n') < (m,n)$,.

By the discussion above we see that $\Theta_{m,n}(\lambda) = D_{m,n} + Y^{(4)}_{m,n}$, $m,n\le N$, $m+n\le k$, are purely imaginary provided that for all pairs $(m',n') < (m,n)$ the functions $R_{m',n'}(\lambda)$ are admissible and $\Theta_{m',n'}(\lambda)$ are purely imaginary.

Recall the definition and property of $Y_{m,n}(\lambda)$ in (54). We observe that $Y_{m,n}(\lambda) = \Theta_{m,n}(\lambda)$ when $m+n\le N$ by the fact that the expansion in the $k \ge N+1$ step does not affect the coefficients of $y^m\bar{y}^n$, $m+n\le N$. □

Now we turn to the proof of the rest of Lemma 7.9, i.e. the claim on the function $f_{m,n}$. We plug the equation (34) into Equation (31), and use that

$$P_c J\vec{N}(\vec{R},z) = \sum_{2\leq m+n\leq k} y^m\bar{y}^n P_c N_{mn}(\lambda) + A_k(y,\bar{y})R_k + N_N(R_N,y,\bar{y}) + F_k(y,\bar{y}),$$

where the terms $A_k(y,\bar{y})$, $N_N(R_N,y,\bar{y})$ and $F_k(y,\bar{y})$ are described in Theorem 7.2, and $k > N$. The result is

$$\frac{d}{dt}R_k = [L(\lambda) + \dot{\gamma}P_c J + \dot{\lambda}P_{c\lambda} + A_k(y,\bar{y})]R_k + \sum_{n=1}^{5} G_n + N_N(R_N,y,\bar{y}) + F_k(y,\bar{y})$$

where

$$G_1 := P_c \sum_{2\leq m+n\leq k} y^m\bar{y}^n[L(\lambda) - i(m-n)\epsilon(\lambda)]R_{m,n}(\lambda);$$

$$G_2 := -P_c \sum_{2\leq m+n\leq k} [\frac{d}{dt}y^m\bar{y}^n - i(m-n)\epsilon(\lambda)y^m\bar{y}^n]R_{m,n}(\lambda);$$

$$G_3 := -P_c\dot{\lambda} \sum_{2\leq m+n\leq k} y^m\bar{y}^n\partial_\lambda R_{m,n}(\lambda);$$

$$G_4 := P_c \sum_{2\leq m+n\leq k} y^m\bar{y}^n(\dot{\lambda}P_{c\lambda} + \dot{\gamma}P_c J)R_{m,n}(\lambda)$$

and

$$G_5 := \frac{1}{2}\dot{\gamma}P_c[z\begin{pmatrix} -i\eta \\ \xi \end{pmatrix} + \bar{z}\begin{pmatrix} i\eta \\ \xi \end{pmatrix}] - \frac{1}{2}\dot{\lambda}P_c[z\begin{pmatrix} \xi_\lambda \\ -i\eta_\lambda \end{pmatrix} + \bar{z}\begin{pmatrix} \xi_\lambda \\ i\eta_\lambda \end{pmatrix}].$$

Plug the expansions for $\dot{y}$, $\dot{\lambda}$ and $\dot{\gamma}$ given in Equations (58)-(60), which are proved in Lemma A.1, into G_l, $l = 3,\cdots,5$, to obtain

$$G_l = \sum_{2\leq m+n\leq k} y^m\bar{y}^n G^{(l)}_{m,n}(\lambda) + F_k(y,\bar{y})$$

where for each $m,n \leq N$ the function $iG^{(l)}_{mn}(\lambda)$ is admissible if $R_{m',n'}(\lambda)$ are admissible for all pairs $(m',n') < (m,n)$. Moreover, if $R_{m',n'}(\lambda)$ are of the forms

$$\prod_k (L(\lambda) - ik\epsilon(\lambda) + 0)^{-n_k} P_c\phi_{m',n'}(\lambda)$$

for all the pairs $(m',n') < (m,n)$ then, using the observation that $P_c\partial_\lambda\prod_k(L(\lambda)-ik\epsilon(\lambda)+0)^{-n_k}P_c\phi_{m',n'}(\lambda)$ and $P_c J\prod_k(L(\lambda)-ik\epsilon(\lambda)+0)^{-n_k}P_c\phi_{m',n'}(\lambda)$ are of the form $\prod_k(L(\lambda)-ik\epsilon(\lambda)+0)^{-n_k}P_c\phi^{(2)}_{m',n'}(\lambda)$, we can show that the functions $G^{(l)}_{m,n}(\lambda)$, $\max\{m,n\} > N$, are of a similar form also.

For G_2, using the equation for y in (58) we have

$$\begin{aligned} G_2 &= \sum_{2\le m'+n'\le k}\sum_{2\le m+n\le k} mR_{mn}(\lambda)\Theta_{m',n'}(\lambda)y^{m-1+m'}\bar{y}^{n+n'} \\ &\quad + \sum_{2\le m'+n'\le k}\sum_{2\le m+n\le k} nR_{m,n}(\lambda)\bar{\Theta}_{m',n'}(\lambda)y^{m+n'}\bar{y}^{n-1+m'} + F_k(y,\bar{y}). \end{aligned}$$

Recall the definition and property of $\Theta_{m,n}(\lambda)$ in (58), we have that if $m-1+m', n+n' \le N$, then

$$\text{either } mR_{m,n}(\lambda)\Theta_{m',n'}(\lambda) = 0 \text{ or } (m,n) < (m-1+m', n+n')$$

(this is the point where we use the property that $\Theta_{m,n}(\lambda) = 0$ for $m,n \le N$ and $m \ne n+1$, where, recall the fact that $\Theta_{m,n}(\lambda) = Y_{m,n}(\lambda)$ if $m+n \le N$ proved in Lemma A.1). Thus

$$G_2 = \sum_{2\le m+n\le k} G^{(2)}_{m,n}(\lambda)y^m\bar{y}^n + F_k(y,\bar{y}),$$

where if $m,n \le N$ and $R_{m',n'}(\lambda)$ are admissible for all the pairs $(m',n') < (m,n)$ then $iG^{(2)}_{m,n}(\lambda)$ is admissible.

This completes the proof of Lemma 7.9.

□

B Transformation of y

In this appendix we prove a result used in the proof of Lemma 8.3.

Proposition B.1. *Let complex and real functions $y(t)$ and $\lambda(t)$ satisfy Equations (39)-(40) and (42), and let $P(y,\bar{y})$ be a polynomial of the form*

$$P(y,\bar{y}) = \sum_{N+1\le m+n\le 2N+1} p_{m,n}(\lambda)y^m\bar{y}^n$$

with the coefficients $p_{m,n}(\lambda)$ real for $m,n \le N$. Define $\beta := y + P(y,\bar{y})$. Then we have

$$\dot{\lambda} = \sum_{2\le m+n\le 2N+1} \Lambda^{(1)}_{m,n}(\lambda)\beta^m\bar{\beta}^n + \mathit{Remainder} \tag{119}$$

and

$$\dot{\beta} = i\epsilon(\lambda)\beta + \sum_{3\le m+n\le 2N+1} \Theta^{(1)}_{m,n}(\lambda)\beta^m\bar{\beta}^n + \mathit{Remainder} \tag{120}$$

where the functions $\Lambda^{(1)}_{m,n}(\lambda)$ and $\Theta^{(1)}_{m,n}(\lambda)$ are purely imaginary for $m,n \le N$; $\Theta^{(1)}_{m,n} = 0$ if $m+n \le N$ and $m \ne n+1$; and the term Remainder stands for a function satisfying (40).

Proof. We invert the relation $\beta := y + P(y, \bar{y})$ to get the expression

$$y = \beta + \sum_{N+1 \leq m+n \leq 2N+1} P^{(2)}_{m,n}(\lambda)\beta^m \bar{\beta}^n + O(|\beta|^{2N+2}). \tag{121}$$

Since $p_{m,n}(\lambda)$ are real for $m, n \leq N$, the coefficients $P^{(2)}_{m,n}(\lambda)$ are real for $m, n \leq N$.

Plug Equation (121) into Equation (42) to obtain

$$\dot{\lambda} = \sum_{2 \leq m+n \leq 2N+1} \Lambda^{(1)}_{m,n}(\lambda)\beta^m \bar{\beta}^n + \mathit{Remainder}.$$

We claim that $\Lambda^{(1)}_{m,n}(\lambda)$ are purely imaginary for $m, n \leq N$. Indeed, we observe that

$$\Lambda^{(1)}_{m,n}(\lambda) = \Lambda_{m,n}(\lambda) + \sum_{\substack{m'+l_1=m+1 \\ n'+l_2=n}} m'\Lambda_{m',n'} P^{(2)}_{l_1,l_2}(\lambda) + \sum_{\substack{m'+l_2=m \\ n'+l_1=n+1}} n'\Lambda_{m',n'} \bar{P}^{(2)}_{l_1,l_2}(\lambda),$$

where, recall, $\Lambda_{m,n}(\lambda)$ are purely imaginary for $m, n \leq N$. Since $l_1 + l_2 \geq N+1$, we have that if $m, n \leq N$, $m' \neq 0$, $m' + l_1 = m+1$ and $n' + l_2 = n$ then $m', n', l_1, l_2 \leq N$. This implies that $\Lambda_{m',n'}(\lambda)$ are purely imaginary and $P^{(2)}_{l_1,l_2}(\lambda)$ are real. Hence $m'\Lambda_{m',n'} P^{(2)}_{l_1,l_2}(\lambda)$ are purely imaginary if $m' \neq 0$ (If $m' = 0$ then $m'\Lambda_{m',n'} P^{(2)}_{l_1,l_2}(\lambda) = 0$). By the same reasoning we prove that $n'\Lambda_{m',n'} \bar{P}^{(2)}_{l_1,l_2}(\lambda)$ is purely imaginary for $m, n \leq N$. These two facts together with $\Lambda_{m,n}(\lambda)$ being purely imaginary for $m, n \leq N$ imply that $\Lambda^{(1)}_{m,n}(\lambda)$ are purely imaginary for $m, n \leq N$. This completes the proof of Equation (119) and its properties.

Now we turn to Equation (120). By Equation (39) we obtain

$$\dot{\beta} \;=\; i\epsilon(\lambda)\beta + \sum_{2 \leq m+n \leq 2N+1} \Theta_{mn}(\lambda) y^m \bar{y}^n + K + \mathit{Remainder}, \tag{122}$$

where the term K is defined as

$$K := \frac{d}{dt} \sum_{N+1 \leq m+n \leq 2N+1} p_{m,n}(\lambda) y^m \bar{y}^n - i\epsilon(\lambda) \sum_{N+1 \leq m+n \leq 2N+1} p_{m,n}(\lambda) y^m \bar{y}^n,$$

and recall, the coefficients $\Theta_{m,n}$ are defined in (39).

Using Equation (39) we obtain

$$K = \sum_{N+1 \leq m+n \leq 2N+1} P_{m,n}(\lambda) y^m \bar{y}^n + \mathit{Remainder}. \tag{123}$$

We show below that $P_{m,n}(\lambda)$ are purely imaginary for $m, n \leq N$; and $P_{m,n}(\lambda) = 0$ for $m + n \leq N$. This fact implies that Equation (122) is of the form

$$\dot{\beta} = i\epsilon(\lambda)\beta + \sum_{2 \leq m+n \leq 2N+1} \Theta^{(2)}_{mn}(\lambda) y^m \bar{y}^n + \mathit{Remainder}, \tag{124}$$

where $\Theta^{(2)}_{m,n}(\lambda)$ are purely imaginary for $m,n\le N$, and $\Theta^{(2)}_{m,n}=0$ if $m+n\le N$ and $m\neq n+1$. Substitute into the right hand side of the expansion for y given by Equation (121) to obtain a new equation for β

$$\begin{aligned}
\dot{\beta} &= i\epsilon(\lambda)\beta+\sum_{2\le m+n\le 2N+1}\Theta^{(2)}_{m,n}(\lambda)\beta^m\bar{\beta}^n\\
&\quad+\sum_{N+1\le m'+n'}\sum_{2\le m+n\le 2N+1} m\Theta^{(2)}_{m,n}(\lambda)P^{(2)}_{m',n'}\lambda\beta^{m+m'-1}\bar{\beta}^{n+n'}\\
&\quad+\sum_{N+1\le m'+n'}\sum_{2\le m+n\le 2N+1} n\Theta^{(2)}_{m,n}(\lambda)\bar{P}^{(2)}_{m',n'}(\lambda)\beta^{m+n'}\bar{\beta}^{n+m'-1}+Remainder\\
&= i\epsilon(\lambda)\beta+\sum_{2\le m+n\le 2N+1}\Theta^{(3)}_{m,n}(\lambda)\beta^m\bar{\beta}^n.
\end{aligned}\tag{125}$$

By the properties of $\Theta^{(2)}_{m,n}$ above and the facts that $P^{(2)}_{m,n}(\lambda)$ are real if $m,n\le N$ and $P^{(2)}_{m,n}(\lambda)=0$ if $m+n\le N$, we have that $\Theta^{(3)}_{m,n}(\lambda)=0$ if $m\neq n+1$ and $m+n\le N$; $\Theta^{(3)}_{m,n}(\lambda)$ is purely imaginary for $m,n\le N$.

To complete the proof of Proposition B.1 it remains to prove the claim above that $P_{m,n}(\lambda)$ are purely imaginary for $m,n\le N$ and $P_{m,n}(\lambda)=0$ for $m+n\le N$. Compute

$$\begin{aligned}
K &= \dot{\lambda}\sum_{N+1\le m+n\le 2N+1}\partial_\lambda p_{m,n}(\lambda)y^m\bar{y}^n\\
&\quad+\sum_{N+1\le m+n\le 2N+1}p_{m,n}(\lambda)[\frac{d}{dt}y^m\bar{y}^n-i\epsilon(\lambda)y^m\bar{y^n}]\\
&= \sum_{2\le m'+n'\le 2N+1,\ N+1\le m+n\le 2N+1}\sum \partial_\lambda p_{m,n}(\lambda)\Lambda_{m',n'}y^{m+m'}\bar{y}^{n+n'}\\
&\quad+\sum_{N+1\le m+n\le 2N+1,\ 2\le m'+n'\le 2N+1}\sum mp_{m,n}(\lambda)\Theta_{m',n'}y^{m-1+m'}\bar{y}^{n+n'}\\
&\quad+\sum_{N+1\le m+n\le 2N+1,\ 2\le m'+n'\le 2N+1}\sum np_{m,n}\bar{\Theta}_{m',n'}(\lambda)y^{m+n'}\bar{y}^{n-1+m'}\\
&\quad+i\epsilon(\lambda)\sum_{N+1\le m+n\le 2N+1}(m-n-1)p_{m,n}(\lambda)y^m\bar{y}^n+Remainder.
\end{aligned}\tag{126}$$

We have that $P_{m,n}(\lambda)=0$ for $m+n\le N$ since all the expressions above are of order $o(|y|^N)$. Next we show that $P_{m,n}(\lambda)$ are purely imaginary for $m,n\le N$. We have the following observations for the four terms on the right hand side of (126)

(A) if $m+m',n+n'\le N$, then we have $m,n,m',n'\le N$ which implies that $\Theta_{m',n'}(\lambda)$ is purely imaginary and $p_{m,n}(\lambda)$ is real. Thus $\partial_\lambda p_{m,n}\Theta_{m',n'}(\lambda)$ is purely imaginary;

(B) if $m-1+m',n+n'\le N$, then either $mp_{m,n}\Theta_{m',n'}(\lambda)$ is zero or $m,n,m',n'\le N$ by the properties of $p_{m,n}(\lambda)$ and $\Theta_{m',n'}(\lambda)$. Thus $mp_{m,n}\Theta_{m',n'}(\lambda)$ is purely imaginary;

(C) if $m+n', n-1+m' \leq N$, then $np_{m,n}\bar{\Theta}_{m',n'}(\lambda)$ is purely imaginary by the same reasoning as in (B) above;

(D) $i\epsilon(\lambda)p_{m,n}(\lambda)$ is purely imaginary for $m, n \leq N$ since the coefficients $p_{m,n}(\lambda)$ are real.

Collecting the results above we conclude that $P_{m,n}(\lambda)$ are purely imaginary for $m, n \leq N$. This completes the proof of the claim made after Equation (123) and, with it, the proof of Proposition B.1. □

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